

ALGEBRA

algebra

A) Function and Function Value

Function

- The relation between dependent and independent variable
- Can be:
 - Explicit
 - $y = f(x)$
 - Dependent variable (y) isolated
 - Implicit
 - $f(x, y) = 0$
 - Dependent variable (y) is not isolated

Example

If $f(x) = 10^x + 1$, find $f(x+1) - f(x)$.

- $G: f(x) = 10^x + 1$
- $R: f(x+1) - f(x) = ?$
- $E, S:$
 - $f(x+1) = 10^{x+1} + 1$
 - $= 10(10^x) + 1$
 - $f(x+1) - f(x)$
 - $= 10(10^x) + 1 - (10^x + 1)$
 - $= 10(10^x) - 10^x$
 - $= 9(10^x)$
- $A:$
 - $f(x+1) - f(x) = 9(10^x)$

Function Value

- Value of function at the given condition/s

Example

If $f = x^2y + xy^2 - 2x + 3y - 5$, find f when $x = -1$ and $y = 2$.

- $G: f(x, y) = x^2y + xy^2 - 2x + 3y - 5$
- $R: f(-1, 2) = ?$
- $E, S:$ calculate in calculator
- $A: f(-1, 2) = 1$

Inverse Function, $f^{-1}(x)$

- Steps in finding inverse function:
 1. Interchange x and y.
 2. Solve for y in terms of x.
 3. Replace y by $f^{-1}(x)$.

Example

Find the inverse function for $f(x) = 2x + 1$

- $G: f(x) = 2x + 1$
- $R: f^{-1}(x) = ?$
- $E, S:$
 - $y = 2x + 1$
 - $x = 2y + 1$
 - $y = \frac{x-1}{2}$
- $A: f^{-1}(x) = \frac{x-1}{2}$

B) Solutions and Properties of Common One-Variable Equations

Quadratic Equation

- $ax^2 + bx + c = 0$
- Solution Methods:
 - Factoring
 - Completing the Square
 - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 - Discriminant, d
 - $b^2 - 4ac$
 - Quadratic Formula
 - ES Calculator
- Properties of Roots
 - Sum
 - $x_1 + x_2 = -\frac{b}{a}$
 - Product
 - $x_1 \cdot x_2 = \frac{c}{a}$

- Cases for Roots:
 - Real and Equal, $x_1 = x_2$
 - $d = 0$
 - Real and Distinct, $x_1 \neq x_2$
 - $d > 0$
 - Complex and Conjugate, $x_{1,2} = \sigma \pm j\omega$
 - $d < 0$

- S:
 - $-\frac{b}{a} = \frac{c}{a}$
 - $b = -c$
 - $2k - 1 = -(-3k + 2)$
 - $k = 1$
- A: $k = 1$

Example

Find the values of x from quadratic equation $x^2 - 5x + 6 = 0$

- G: $a = 1; b = -5; c = 6$
- R: $x_{1,2} = ?$
- S: calculator
- A: $x_1 = 3; x_2 = 2$

Example

Find the sum and product of the roots without solving the equations.

- G: $a = 1; b = -5; c = 6$
- R: $x_1 + x_2 = ?; x_1 x_2 = ?$
- E:
 - $x_1 + x_2 = -\frac{b}{a}$
 - $x_1 x_2 = \frac{c}{a}$
- S: substituting...
- A: $x_1 + x_2 = 5; x_1 x_2 = 6$

Example

Find the value of k from the quadratic equation $7x^2 + (2k - 1)x - 3k + 2 = 0$ so that the sum and product of the roots are equal

- G:
 - $a = 7$
 - $b = 2k - 1$
 - $c = -3k + 2$
 - $x_1 + x_2 = x_1 x_2$
- R: $k = ?$
- E:
 - $x_1 + x_2 = -\frac{b}{a}$
 - $x_1 x_2 = \frac{c}{a}$

Cubic Equation

- $ax^3 + bx^2 + cx + d = 0$
- Solution Methods
 - Synthetic Division
 - Cardan(o)'s Formula
 - Calculator
- Properties of Roots
 - Sum
 - $x_1 + x_2 + x_3 = -\frac{b}{a}$
 - Product
 - $x_1 x_2 x_3 = -\frac{d}{a}$
 - Sum of Products
 - $x_1 x_2 + x_1 x_3 + x_2 x_3 = \frac{c}{a}$

Example

Find the roots of the equation.

$$x^3 - 13x + 12 = 0$$

- G: the eq.
- R: $x_1, x_2, x_3 = ?$
- E, S: calculator
- A:
 - $x_1 = -4$
 - $x_2 = 3$
 - $x_3 = 1$

Example

Find the sum, product, and sum of product of roots without solving for the roots.

- G: $a = 1; b = 0; c = -13; d = 12$
- R: $x_1 + x_2 = ?; x_1 x_2 = ?; x_1 x_2 + x_1 x_3 + x_2 x_3 = ?$
- E:
 - $x_1 + x_2 = -\frac{b}{a}$

- $x_1x_2 = -\frac{d}{a}$
- $x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}$
- S: substituting...
- A:
 - $x_1 + x_2 = 0$
 - $x_1x_2 = -12$
 - $x_1x_2 + x_1x_3 + x_2x_3 = -13$

Roots of Properties for n-degree Equations

- $ax^n + bx^{n-1} + \dots + yx + z = 0$
- Sum of all roots:
 - $x_1 + x_2 + \dots + x_n = -\frac{b}{a}$
- Sum of product (choose two)
 - $x_1x_2 + \dots + x_{n-1}x_n = \frac{c}{a}$
 - For quadratic equation:
 - $x_1x_2 = \frac{c}{a}$
 - For cubic equation:
 - $x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}$
- Sum of product (choose three)
 - $x_1x_2x_3 + \dots + x_{n-2}x_{n-1}x_n = -\frac{d}{a}$
 - For cubic equation:
 - $x_1x_2x_3 = -\frac{d}{a}$
- Sum of product (choose four)
 - $x_1x_2x_3x_4 + \dots + x_{n-3}x_{n-2}x_{n-1}x_n = \frac{e}{a}$
 - For fourth degree equation:
 - $x_1x_2x_3x_4 = \frac{e}{a}$
- Remembering signs:
 - $a = +$
 - $b = -$
 - $c = +$
 - $d = -$
 - $e = +$

Quadratic Equation

Sum:

$$\bullet x_1 + x_2 = -\frac{b}{a}$$

Product:

$$\bullet x_1x_2 = \frac{c}{a}$$

-

Cubic Equation

Sum:

$$\bullet x_1 + x_2 + x_3 = -\frac{b}{a}$$

Sum of Product:

$$\bullet x_1x_2 + x_1x_3 + x_2x_3 = \frac{c}{a}$$

Product:

$$\bullet x_1x_2x_3 = -\frac{d}{a}$$

Given: $ax^n + bx^{n-1} + \dots + yx + z = 0$

Property	Formula
Sum of all roots	$x_1 + x_2 + \dots + x_n = -\frac{b}{a}$
Sum of product (choose two)	$x_1x_2 + \dots + x_{n-1}x_n = \frac{c}{a}$
Sum of product (choose three)	$x_1x_2x_3 + \dots + x_{n-2}x_{n-1}x_n = -\frac{d}{a}$
Sum of product (choose four)	$x_1x_2x_3x_4 + \dots + x_{n-3}x_{n-2}x_{n-1}x_n = \frac{e}{a}$
...	...

C) Rene Descartes' Forecast of Roots of a One-Variable Equation

Forecast of Roots

- Model: $f(x) = 0$
- No. of roots = Degree of $f(x)$
- No. of + roots = No. of change in signs in $f(x)$ minus an even no.
- No. of - roots = No. of change in signs in $f(-x)$ minus an even no.

Example

How many real positive roots are there in the equation.

$$x^4 - 4x^3 + 4x^2 - 6x - 18 = 0$$

- G: the eq'n
- R: $n = \text{no. of real positive roots}$
- E, S:
 - No. of change in signs in $f(x) = 3$
 - $n = 3 - 0 \text{ or } 3 - 2$

- A:
 - $n = 3 \text{ or } 1$

D) Common Solutions to System of Linear Equations

2 unknowns

- Solution Methods:
 - Elimination
 - Substitution
 - Calculator

Example

Find the values of x and y from the system of linear equations:

$$\begin{aligned} 2x + y &= 4 \\ x - 2y &= -3 \end{aligned}$$

- G: SLE
- R: $x = ? ; y = ?$
- E, S: calculator
- A: $x = 1 ; y = 2$

3 Unknowns

- Solution Methods:
 - Elimination
 - Cramer's Rule (Determinants)
 - Calculator

Example

Find the values of x, y, and z from the system of linear equations.

$$\begin{aligned} 2x + y - 2z &= -2 \\ x + 2y + z &= 8 \\ -x + 3y + z &= 8 \end{aligned}$$

- G: SLE
- R: $x = ? ; y = ? ; z = ?$
- E, S: calculator
- A: $x = 1 ; y = 2 ; z = 3$

E) The Remainder Theorem

Remainder Theorem

- Theorem for determining constant remainder R when dividing a polynomial $f(x)$ by a first-order binomial $x - r$
- $\frac{f(x)}{x-r} = q(x) + \frac{R}{x-r}$
 - $R = f(r)$

Factor Theorem

- If $R = f(r) = 0$, then:
 - $(x - r)$ is a factor of polynomial $f(x)$
 - r is a root of function $f(x)$

Example

Find the constant remainder when $x^2 - 3x + 5$ is divided by $x + 2$

- G: $\frac{x^2-3x+5}{x+2}; r = -2; f(x) = x^2 - 3x + 5$
- R: $R = \text{remainder} = ?$
- E: $R = f(r)$
- S: calculate $f(-2)$ in calculator.
- A: $R = 15$

F) The Binomial Theorem

Binomial Theorem

- Theorem of expanding binomial $(a + b)^n$
- $r^{\text{th}} \text{ term} = (nCq)a^p b^q$
 - $q = r - 1$
 - $p + q = n \rightarrow p = n - q$

Other Properties in Binomial Expansion

- First term is a^n
- Last term is b^n
- Sum of Exponents, SOE
 - $(a + b)^n \rightarrow SOE = n(n + 1)$
 - $(a + k)^n \rightarrow SOE = \frac{n(n+1)}{2}$
 - $(a^x + b^y) \rightarrow SOE = \frac{n(n+1)(x+y)}{2}$
- Sum of Numerical Coefficients, SNC
 - $(a + b)^n$

- Replace variables by 1.
- Proceed with operations.
- $(2x + 3y)^5$
 - $SNC = (2(1) + 3(1))^5$
 - $SNC = 3125$
- $(a + k)^n$
 - Replace variables by 1.
 - Proceed with operations.
 - Subtract k^n
 - $(2x + 7)^3$
 - $SNC = (2(1) + 7)^3$
 - $SNC = 729$
 - $SNC = 729 - 7^3$
 - $SNC = 386$

Example

Find the middle term without expansion of the binomial $(x + 2y)^{10}$.

- $G: (x + 2y)^{10} \rightarrow (a + b)^n$
- $R: a = \text{middle term} = ?$
- $E:$
 - $r^{\text{th}} \text{ term} = (nCq)a^p b^q$
 - $q = r - 1$
 - $p = n - q$
- $S:$
 - $r = \frac{n+2}{2} = 6$
 - $q = 5$
 - $p = 5$
 - $a = (10C5)x^5(2y)^5$
 - $a = 252(2)^5 x^5 y^5$
 - $a = 8064x^5 y^5$
- $A: a = 8064x^5 y^5$

Example

Find the numerical coefficient NC of the 5th term w/o expansion of the binomial $(x + 4y)^{12}$

- $G: (x + 4y)^{12} \rightarrow (a + b)^n$
- $R: a_5 = 5^{\text{th}} \text{ term} = kf(x, y); k = ?; r = 5$
- $E: r^{\text{th}} \text{ term} = (nCq)a^p b^q$
 - $q = r - 1$

- $p = n - q$
- $S:$
 - $q = 4$
 - $p = 8$
 - $a_5 = (12C4)x^8(4y)^4$
 - $k = (12C4)(4)^4$
 - $k = 126720$
- $A: k = 126720$

Examples

Find the term without x in the expansion of $(x^2 - \frac{1}{x})^9$.

- $G: (x^2 - \frac{1}{x})^9 \rightarrow (a + b)^n$
- $R: a = \text{term w/o } x = ?$
- $E: r^{\text{th}} \text{ term} = (nCq)a^p b^q$
- $S:$
 - $a^p b^q = x^0$
 - $x^{2p} \cdot -\frac{1}{x^q} = x^0$
 - $2p - q = 0$
 - $p + q = 9$
 - $3p = 9$
 - $p = 3$
 - $q = 6$
 - $r = 7$
 - $a = (9C6)(x^2)^3 \left(-\frac{1}{x}\right)^6$
 - $a = 84$
- $A: a = 84$

F.a) Common Number Series or Progression

Series of Perfect Squares

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Example

Identical spheres are piled in the form of pyramid w/ square base until one sphere on top. Find the total no. of spheres if there are four spheres on each side of square base.

- $G: n = 4$
- $R: x = \text{no. of spheres}$

Sum of Perfect Squares

$$1^2 + 2^2 + 3^2 + \dots + n^2$$

$$S_n = \frac{n(n+1)(2n+1)}{6}$$

Sum of Sums

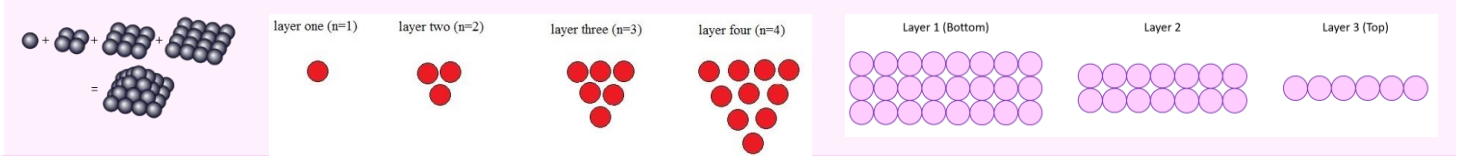
$$(1) + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n)$$

$$S_n = \frac{n(n+1)(n+2)}{6}$$

Sum of Products

$$(n)(m) + (n-1)(m-1) + (n-2)(m-2) + \dots + (1)(m-n+1)$$

$$S_n = \frac{n(n+1)(3m-n+1)}{6}$$



- $E: x = 1^2 + 2^2 + 3^2 + 4^2 = \frac{n(n+1)(2n+1)}{6} \Big|_{n=4}$
- S : Evaluating x
- $A: x = 30$

Sum of Sums

- $(1) + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n) = \frac{n(n+1)(n+2)}{6}$
- Similar to stacking spheres to form a pyramid with an equilateral triangle as base:
 - The top would have 1 sphere..
 - The layer below has 1+2=3 sphere.
 - The layer below has 1+2+3=6 sphere.
 - And so on.

Sum of Products

- $(n)(m) + (n-1)(m-1) + (n-2)(m-2) + \dots + (1)(m-n+1) = \frac{n(n+1)(3m-n+1)}{6}$
- $n < m$
- $m = 8, n = 3$
 - $(3)(8) + (2)(7) + (1)(6)$
- Similar to Stacking Spheres to form a pyramid with a rectangular base:
 - The base may have 3 by 8 spheres.
 - The layer above will have 2 by 7 spheres.
 - The layer above will have 1 by 7 spheres.
 - Any layer has one less sphere by length and width than the layer below.

F.b) Arithmetic Sequence

Arithmetic Sequence

- Set of numbers with common difference
- $a_1, a_2, a_3, \dots, a_n$
- Formulas:
 - $d = a_2 - a_1 = a_3 - a_2 = \dots$
 - $d = \text{common difference}$
 - $a_n = a_1 + (n-1)d$
 - $a_n = \text{nth term}$
 - $S_n = \frac{n}{2}(a_1 + a_n)$
 - $S_n = \text{sum of } n \text{ terms}$
 - $S_n = \frac{n}{2}(2a_1 + (n-1)d)$
 - $AM = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$
 - $AM = \text{Arithmetic Mean}$
 - $AM = \frac{x+y}{2}$ for two nos. x and y .

Example

In the pile of logs, each layer contains one more log than the layer above and the top contains just one log. How many layers are there if there are 105 logs in the pile.

- $G: S_n = 105; a_1 = 1; d = 1$
- $R: n = ?$
- $E: S_n = \frac{n}{2}(2a_1 + (n-1)d)$
- $S: 105 = \frac{n}{2}(2(1) + (n-1)(1))$
- $A: n = 14$
- (another sol'n)
- $G: 1 + 2 + 3 + \dots + n = 105$
- $R: ?$

- $E, S: S_n = \frac{n}{2}(2a_1 + (n-1)d)(1) \rightarrow$
 $y = A + Bx + Cx^2$
- S : Quadratic or Parabolic Regression in Calculator

x (no. of terms)	y (sum)
1	1
2	1+2=3
3	1+2+3=6
$\hat{x}$	105

$$105\hat{x} \rightarrow 14$$

- $A: n = 14$

Regression Solution Methods

- Needed data is $n+1$ rows where n is the degree.
- Linear Regression has degree one and needs two rows of data.
- Quadratic Regression has degree two and needs three rows of data.

Example

For the interval 84 to 719 or $[84, 719]$, find:

- The no. of integers which are divisible by 5.
- The sum of nos. in (a).

- $G: [84, 719], \text{divisible by } 5$

- $R: n = ?; S_n = ?;$

- E :

- $a_n = a_1 + (n-1)d$
- $S_n = \frac{n}{2}(a_1 + a_n)$

- S :

- 85, 90, 95, ..., 715
- $a_1 = 85, a_n = 715, d = 5$
- $715 = 85 + (n-1)5$
- $n = 127$
- $S_{127} = \frac{127}{2}(85 + 715)$
- $S_{127} = 50800$

- A :

- $n = 127$
- $S_{127} = 50800$

- (another sol'n)

- $G: [84, 719], \text{divisible by } 5$

- $R: n = ?; S_n = ?;$

- E, S :

- $a_n = a_1 + (n-1)d \rightarrow$
 $y = Bx + A$

Linear Regression in Calculator

x (n)	y (nth term)
1	85
2	90
$\hat{x}$	715

$$715\hat{x} \rightarrow 127 = n$$

- $S_n = \frac{n}{2}(2a_1 + (n-1)d) \rightarrow$
 $y = A + Bx + Cx^2$

Quadratic Regression in Calculator

x (n)	y (nth term)
1	85
2	85+90=175
3	85+90+95=270
127	$\hat{y}$

$$127\hat{y} \rightarrow 50800 = S_n$$

- A :

- $n = 127$
- $S_n = 50800$

F.c) Geometric Sequence

Geometric Sequence

- It deals with nos. of terms w/ common ratio.
- $a_1, a_2, a_3, \dots, a_n$
- Formulas for finite geometric seq.:
 - $r = \frac{a_2}{a_1} = \frac{a_3}{a_2} = \dots$
 - $r = \text{common ratio}$
 - $a_n = a_1 r^{(n-1)}$
 - $a_n = \text{nth term}$
 - $S_n = a_1 \left(\frac{1-r^n}{1-r} \right)$
 - $S_n = \text{sum of } n \text{ terms}$
 - $GM = \sqrt[n]{a_1 a_2 a_3 \dots a_n}$
 - $GM = \text{geometric mean}$

- For two nos. x and y

- $GM = \sqrt{xy}$

- Formulas for infinite geometric seq.:

- $S = \frac{a_1}{1-r}$

Example

Find the sum of the first 5 terms in a geometric progression if the third term is 144 and the sixth term is 486.

- $G: a_3 = 144; a_6 = 486$

- $R: S_5 = ?$

- $E:$

- $S_n = a_1 \left(\frac{1-r^n}{1-r} \right)$

- $a_n = a_1 r^{(n-1)}$

- $S:$

- $a_3 = a_1 r^2 = 144$

- $a_6 = a_1 r^5 = 486$

- $\frac{a_6}{a_3} = \frac{r^5}{r^2} = \frac{486}{144}$

- $r^3 = 3.375$

- $r = 1.5$

- $a_1 = \frac{144}{1.5^2} = 64$

- $S_5 = 64 \left(\frac{1-1.5^5}{1-1.5} \right)$

- $A: S_5 = 844$

- (another sol'n)

- $G: a_3 = 144; a_6 = 486$

- $R: S_5 = ?$

- $E, S:$

- $a_n = a_1 r^{(n-1)} \rightarrow$
 $y = AB^x$

Exponential Regression in Calculator

x (n)	y (nth term)
1	$\hat{y}$
3	144
6	486

$$1\hat{y} \rightarrow 64 = a_1$$

- Input in calculator:

- $\Sigma_{n=1}^5 (64(1.5)^{(n-1)})$

- $A: S_5 = 844$

Example

A rubber ball is dropped from a height of 18 m. If it rebounds up to $\frac{2}{3}$ of the height from where it last fell and so on. Find the total distance transverse by the ball until it stop.

- $G: h_0 = 18m$

- $R: d = \text{total distance}$

- $E: S_n = a_1 \left(\frac{1-r^n}{1-r} \right)$

- $S:$

- Downward distance starts at 18m

- $d_{\text{down}} = 18 \left(\frac{1}{1-\frac{2}{3}} \right)$

- $d_{\text{down}} = 54$

- Upward distance starts at $\frac{2}{3}$ of 18m = 12m when ball hits ground after drop

- $d_{\text{up}} = 12 \left(\frac{1}{1-\frac{2}{3}} \right)$

- $d_{\text{up}} = 36$

- $d = d_{\text{up}} + d_{\text{down}}$

- $A: d = 90m$

Ball Dropped From a Height Problem Shortcut

- $d = h_0 + \frac{2rh_0}{1-r} = h_0 + \frac{2h_1}{1-r}$

- Solving previous problem

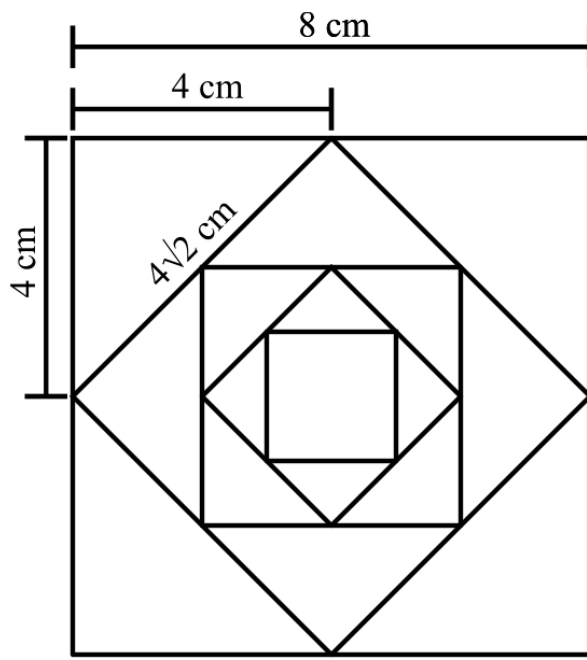
- $d = 18 + \frac{2\left(\frac{2}{3}\right)(18)}{1-\frac{2}{3}}$

- $d = 90m$

Example

Given is an 8 cm square. If the second square is made by connecting the midpoints of the sides of the first square and the third square is made by connecting the midpoints of the sides of the second square, and continuous indefinitely. Find the sum of the perimeter of the squares.

• G:



• R: $\Sigma P = ?$

• E: $S_n = a_1 \left(\frac{1}{1-r} \right)$

• S:

- $P_1 = 4(8) = 32$
- $P_2 = 4(4\sqrt{2}) = 16\sqrt{2}$
- $P_3 = 4(4) = 16$
- $r = \frac{16\sqrt{2}}{32} = \frac{16}{16\sqrt{2}} = \frac{\sqrt{2}}{2}$
- Subs. In $\Sigma P = P_1 \left(\frac{1}{1-r} \right)$

• A: $\Sigma P = 109.25 \text{ cm}$

F.d) Harmonic Sequence

Harmonic Sequence

- Set of numbers or terms whose reciprocals form an arithmetic sequence
- $A_1, A_2, A_3, \dots, A_n$
- Formula:
 - Terms:
 - $a_1 = \frac{1}{A_1}$
 - $a_2 = \frac{1}{A_2}$
 - ...
 - $HM = \frac{n}{\frac{1}{A_1} + \frac{1}{A_2} + \dots + \frac{1}{A_n}}$

$$HM = n \left(\frac{1}{A_1} + \frac{1}{A_2} + \dots + \frac{1}{A_n} \right)^{-1}$$

- $HM = \text{Harmonic Mean}$
- For two nos. x and y , $H = \frac{2xy}{x+y}$

Example

The 3rd term in a harmonic progression is 15 and the 9th term is 6. Find the eleventh term.

- G: $A_3 = 15; A_9 = 6$
- R: $A_{11} = ?$
- E: $a_n = \frac{1}{A_n}$
- S:
 - $a_3 = \frac{1}{15}$
 - $a_9 = \frac{1}{6}$
 - $a_3 = a_1 + 2d = \frac{1}{15}$
 - $a_9 = a_1 + 8d = \frac{1}{6}$
 - $a_9 - a_3 = 6d = \frac{1}{10}$
 - $d = \frac{1}{60}$
 - $a_1 = \frac{1}{15} - 2 \left(\frac{1}{60} \right) = \frac{1}{30}$
 - $a_{11} = \frac{1}{30} + 10 \left(\frac{1}{60} \right)$
 - $a_{11} = \frac{1}{5}$
- A: $A_{11} = 5$

G) Motion Problems

Constant Velocity

- Formula: $s = vt$
 - $\text{distance} = \text{velocity} \times \text{time}$
- 2 Situations:
 - Body Moving in Stationary Medium
 - $v = \text{ave. or constant speed}$
 - Body Moving (with velocity x) in Moving Medium (with velocity y)
 - Body Going with Medium
 - $v = x + y$
 - Body Going against Medium
 - $v = x - y$

- 2 Bodies Moving in Stationary Medium
 - $s = ut$
 - $s =$
distance between two bodies
 - $u =$ *relative velocity*
 - Moving in opposite directions
 - $u = v_{fast} + v_{slow}$
 - Moving in the same direction
 - $u = v_{fast} - v_{slow}$
- Steps for Solution:
 - Draw figure. Write labels.
 - Set-up equations by considering
 - Distance Analysis
 - Time Analysis
 - Use $s = vt$

Example

A bus leaves Manila at 12 noon for Baguio 250 km away, travelling at an average speed of 55 kph. At the same time, another bus leaves Baguio for Manila traveling at an average speed of 65 kph. At what time will they meet?

- G :
 - $t = 0@12NN$
 - $s = 250km$
 - $x = 55kph$
 - $y = 65kph$
- $R: T = \text{time of meeting} = ?$
- E, S :
 - $v = x + y$
 - $v = 120$
 - $s = vt$
 - $t = \frac{s}{v} = \frac{250}{120} = 2.0833 \text{ h} = 2h5m$
- $A: T = 2:05PM$

Example

A 4m long sports car wants to overtake a trailer truck 50 m long and traveling at 72 kph in 3 seconds. Find the speed needed for the sports car.

- G :
 - $s = 50 + 4 = 54 \text{ m}$
 - $v_{slow} = 72 \frac{km}{hr} \times \frac{1000m}{1km} \times \frac{1hr}{60 \cdot 60s} = 20m/s$

- $t = 3$
- $R: v = ?$
- E, S :
 - $u = v - 72$
 - $s = ut$
 - $54 = (v - 20)3$
- $A: v = 38 \text{ m/s} = 136.8 \text{ kph}$

Example

A boatman can travel 10 km downstream. At the same time, he can travel 6 km upstream. If the water current is 3 kph, find the speed of the boat in still water.

- G :
 - Downstream:
 - $s_{down} = 10kph$
 - Upstream:
 - $s_{up} = 6km$
 - $t_{down} = t_{up}$
 - $y = 3 \text{ kph}$
- $R: x = \text{speed of boat in still water} = ?$
- E, S :
 - Downstream:
 - $v_{down} = x + 3$
 - Upstream:
 - $v_{up} = x - 3$
 - Doing time analysis...
 - $t_{up} = t_{down}$
 - $\frac{s_{up}}{v_{up}} = \frac{s_{down}}{v_{down}}$
 - $\frac{6}{x-3} = \frac{10}{x+3}$
 - $x = 12$
- $A: x = 12 \text{ kph}$

Example

Chuck on his bicycle intends to arrive at a certain time to a town that is 30 km away from his home. After riding 10 km, he rested for half an hour, and as a result, he was obliged to ride the rest of the trip 2 kph faster. What was his original speed.

- G :
 - $s = 30km$
 - $s_1 = 10km$
 - $s_2 = 20km$

- $t_{rest} = 0.5 h$
- $v_2 = v_1 + 2$
- $v_1 = v$
- $R: v = ?$
- $E: t = t_1 + t_{rest} + t_2$
- $S:$
 - $s = vt$
 - $t = \frac{v}{s}$
 - $t = t_1 + t_{rest} + t_2$
 - $\frac{s}{v} = \frac{s_1}{v_1} + t_{rest} + \frac{s_2}{v_2}$
 - $\frac{30}{v} = \frac{10}{v} + 0.5 + \frac{20}{v+2}$
- $A: v = 8 \text{ kph}$

H) Work or Accomplishment Problem

Work or Accomplishment Problem

- General Formulas:
 - For a worker:
 - $A = rt$
 - $A = \text{accomplishment}$
 - $r = \text{rate of doing work}$
 - $t = \text{time}$
 - For a group of N workers:
 - $A = rNt$
- Steps for Solutions:
 - Always identify r .
 - Set-up equations by considering accomplishment balance.
 - Sum of Accomplishments = Work to be done
- For a **piece** of work:
 - Case 1: Workers A, B, ... works alone and later work together.
 - $t_{AB...} = \left(\frac{1}{t_A} + \frac{1}{t_B} + \dots \right)^{-1}$
 - Case 2: Workers A, B, and C work in combinations AB, AC, and BC, and later work together.
 - $t_{ABC} = 2 \left(\frac{1}{t_{AB}} + \frac{1}{t_{AC}} + \frac{1}{t_{BC}} \right)^{-1}$

Example

Tom and Gerry are bricklayers. Tom can lay bricks for a wall in 5 days. With Gerry's help, he can build it in 2 days. How long would it take Gerry to build it alone?

- $G:$
 - $A \rightarrow \text{Tom}$
 - $B \rightarrow \text{Gerry}$
 - $t_A = 5 \text{ days}$
 - $t_{AB} = 2 \text{ days}$
- $R: t_B = ?$
- $E: t_{AB} = \left(\frac{1}{t_A} + \frac{1}{t_B} \right)^{-1}$
- $S: \text{Shift-solve.}$
- $A: t_B = 3.33 \text{ days or } 3\frac{1}{3} \text{ days}$

Example

An inlet pipe can fill a tank in 6 hrs. Another inlet pipe can fill the same tank in 10 hrs. If an outlet pipe can empty the tank in 8 hrs. How long will it take to fill the tank when all pipes are open.

- $G:$
 - $t_A = 6 \text{ hrs}$
 - $t_B = 10 \text{ hrs}$
 - $t_C = -8 \text{ hrs}$
- $R: t_{ABC} = \text{time to fill} = ?$
- $E: t_{ABC} = \left(\frac{1}{t_A} + \frac{1}{t_B} + \frac{1}{t_C} \right)^{-1}$
- $S: \text{Substitute.}$
- $A: 7.06 \text{ hrs} = 7 \text{ h } 3 \text{ m } 31.76 \text{ s}$

Example

A and B working together can do a job in 5 days, B and C together can do the same job in 4 days, and A and C in 2.5 days. In how many days can all of them finish the job working together?

- $G:$
 - $t_{AB} = 5 \text{ days}$
 - $t_{BC} = 4 \text{ days}$
 - $t_{AC} = 2.5 \text{ days}$
- $R: t_{ABC} = ?$
- $E: t_{ABC} = 2 \left(\frac{1}{t_{AB}} + \frac{1}{t_{BC}} + \frac{1}{t_{AC}} \right)^{-1}$
- $S: \text{Substitute.}$
- $A: 2.35 \text{ days or } 2 \text{ d } 8 \text{ h } 28 \text{ m } 14.12 \text{ s}$

Example

Jam can paint a car in 8 hrs. James can paint the same car in 6 hrs. They start to paint the car together. After 2 hrs., Jam leaves for lunch and James finishes painting the car alone. How long does it take James to finish?

- *G*:
 - $A \rightarrow \text{Jam}$
 - $B \rightarrow \text{James}$
 - $t_A = 8 \text{ hrs}$
 - $t_B = 6 \text{ hrs}$
 - $t_1 = 2 \text{ hrs}$
 - $A = 1 \text{ painted car}$
- *R*:
 - $t_2 = ?$
- *E*: $A = rt$
- *S*:
 - $r = \frac{A}{t}$
 - $r_A = \frac{1}{8}, r_B = \frac{1}{6}$
 - Work balance: Sum of Accomplishment = Total Accomplishment
 - $A_{AB, 2 \text{ hrs}} + A_B = A$
 - $(r_A + r_B)t_1 + r_B t_2 = A$
 - $\left(\frac{1}{8} + \frac{1}{6}\right)2 + \frac{1}{6}t_2 = 1$
 - Shift-solve.
- *A*: $t_2 = 2.5 \text{ hrs or } 2 \text{ h } 30 \text{ m}$

Examples

Eight men can dig 150 m of trench in 7 days while 3 men can backfill 100 m of trench in 4 days. How long will it take for 10 men to dig and backfill 200 m of trench?

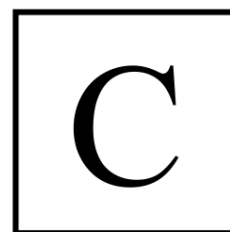
- *G*:
 - $d \rightarrow \text{dig}$
 - $b \rightarrow \text{backfill}$
 - Digging:
 - $A_d = 150 \text{ m}$
 - $t_d = 7 \text{ days}$
 - $N_d = 8 \text{ men}$
 - Backfilling:
 - $A_b = 100 \text{ m}$
 - $t_b = 4 \text{ days}$
 - $N_b = 3 \text{ men}$

- *R*:
 - $A'_d = A'_b = 200 \text{ m}$
 - $N = 10 \text{ men}$
 - $t = ?$
- *E*:
 - $A = Nrt$
- *S*:
 - $150 = 8(r_d)7$
 - $r_d = \frac{75}{28}$
 - $100 = 3r_b(4)$
 - $r_b = \frac{25}{3}$
 - $t = \frac{A'_d}{Nr_d} + \frac{A'_b}{Nr_b}$
 - $t = \frac{200}{10\left(\frac{75}{28}\right)} + \frac{200}{10\left(\frac{25}{3}\right)}$
 - $t = 9.87 \text{ days}$
- *A*: $t = 9.87 \text{ days or } 9 \text{ d } 20 \text{ h } 48 \text{ m}$

I) Mixture Problems

Mixture Problems

X



- Formula:
 - $y = Cx$

y	C	x
Amount of Salt	Concentration of Salt in Liquid (g/mL)	Total Amount of Liquid (mL)
Total amount of something (g, mL)	Percent Composition of Something in a Material (%)	Total Amount of Material (g, mL)

Total Price (Peso)	Cost of Component (Peso / kg)	Component Mass (kg)
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- Steps for Solution:
 - Represent each mixture by a rectangle with a C and x.
 - Set-up equations by balancing materials or cost.
 - Sum of individual costs = Total cost

Example

Given is 100 grams of 50% gold alloy. How many grams of pure gold must be added to obtain an alloy which is 75% gold?

- G:

$$\begin{array}{c} 100 \text{ g} \\ \boxed{\begin{array}{c} 50\% \\ \text{Au} \end{array}} \end{array} + \begin{array}{c} x \text{ g} \\ \boxed{\begin{array}{c} 100\% \\ \text{Au} \end{array}} = \begin{array}{c} (100+x) \text{ g} \\ \boxed{\begin{array}{c} 75\% \\ \text{Au} \end{array}}$$

- R: $x = ?$
- E: $y = Cx$
- S:
 - $(0.5)(100) + x = (0.75)(100 + x)$
 - Shift-solve.
- A:
 - $x = 100 \text{ g}$

Example

Given is 100 L of brine which is 20% salt. How many liters of water must be evaporated to obtain a solution which is 25% salt.

- G:

$$\begin{array}{c} 100 \text{ L} \\ \boxed{\begin{array}{c} 20\% \\ \text{Salt} \end{array}} \end{array} - \begin{array}{c} x \text{ L} \\ \boxed{\begin{array}{c} 0\% \\ \text{Salt} \end{array}} = \begin{array}{c} (100-x) \text{ L} \\ \boxed{\begin{array}{c} 25\% \\ \text{Salt} \end{array}}$$

- R: $x = ?$
- E: $y = Cx$
- S:
 - $(0.2)(100) - 0(x) = (0.25)(100 - x)$
 - Shift-solve.

- A: $x = 20 \text{ L}$

Example

A merchant wishes to reduce the price of her fresh ground coffee by mixing 2 grades. If she has 50 kg of coffee which sells for 500 Php/kg, how much coffee worth 300 Php/kg must she mix with it so that she can sell the final mixture for 425 Php/kg.

- G:

$$\begin{array}{c} 50 \text{ kg} \\ \boxed{\begin{array}{c} 500 \\ \text{Php/kg} \end{array}} \end{array} + \begin{array}{c} x \text{ kg} \\ \boxed{\begin{array}{c} 300 \\ \text{Php/kg} \end{array}} = \begin{array}{c} (x+50) \text{ kg} \\ \boxed{\begin{array}{c} 425 \\ \text{Php/kg} \end{array}}$$

- R: $x = ?$
- E: $y = Cx$
- S:
 - $500(50) + 300(x) = 425(x + 50)$
- A: $x = 30 \text{ kg}$

Coin Problems

- 1\$ = 100¢
- US Coins Denominations
 - 1 penny = 1 ¢
 - 1 nickel = 5 ¢
 - 1 dime = 10 ¢
 - 1 quarter = 25 ¢
 - 1 half-dollar = 50 ¢
- Steps for Solution:
 - Create variable representations for no. of pieces.
 - Ex: Let: x = no. of dimes.
 - Set-up equations considering:
 - No. of Pieces Relations
 - Costs of Value Analysis
- Number of coins should be positive integers.

Example

Jam opens her coin purse, and she finds dimes, quarters, and nickels with a total value of \$1.90. There are twice as many dimes as there are quarters and half as many nickels as quarters. How many quarters are there?

- **G:**
 - $T = 190 \text{ cents}$
 - $d = 2q$
 - $n = \frac{1}{2}q$
- **R:** $q = ?$
- **E, S:**
 - $d(10) + q(25) + n(5) = 190$
 - $2q(10) + q(25) + \frac{1}{2}q(5) = 190$
 - Shift-solve.
- **A:** $q = 4$

Example

James bought \$21.44 of stamps. He bought 10 more 4¢ stamps than 19¢ stamps. The number of 32¢ stamps was 3 times the number of 19¢ stamps. He also bought two \$1 stamps. How many of each kind of stamps did he purchase?

- **G:**
 - $n \rightarrow 19 \text{ ¢ stamps}$
 - $f \rightarrow 4 \text{ ¢ stamps}$
 - $t \rightarrow 32 \text{ ¢ stamps}$
 - Two \$1 stamps
 - $f = n + 10$
 - $t = 3n$
- **R:**
 - $n, f, t = ?$
- **E, S:**
 - $19n + 4f + 32t + 200 = 2144$
 - $19n + 4f + 32t = 1944$
 - $n - f = -10$
 - $3n - t = 0$
 - Input SLE in calculator.
- **A:**
 - $n = 16$
 - $f = 26$
 - $t = 48$

J) Age problems

Age Problems

	Past (x years before present)	Present	Future (y years after present)
Word markers	Was, were, ago	Is, are, now	Will, shall, hence
Age (A) Representation	$A - x$	A	$A + x$

- Steps for solution:
 - Represent present ages with a variable.
 - Ex. Let: B = Bimbi's Present Age
 - Identify tenses of linking verbs.
 - Tabulate ages as per tenses of linking verbs.
 - Set-up equations while considering the linking verbs.

Example

Irene is twice as old as her daughter. Ten years ago, the sum of their ages was 46 years. How old is Irene?

- **G:**

	Past ("ago")	Present ("is")
Irene	I-10	I
Daughter	D-10	D

- $I = 2D$
- $(I - 10) + (D - 10) = 46$
- **R:** $I = ?$
- **E, S:**
 - $I - 2D = 0 \rightarrow \text{eqn (1)}$
 - $I + D = 66 \rightarrow \text{eqn (2)}$
 - $(1) + 2 \times (2)$
 - $I - 2D + 2I + 2D = 0 + 2(66)$
 - $3I = 132$
 - $I = 44$
- **A:** $I = 44$

Example

In a certain family, the sum of parent's ages is twice the sum of children's ages. Five years ago, the sum of parent's ages was 4 times the sum of children's ages during that time. In 15 years, the sum of parent's ages will be equal to the sum of children's ages. How many children are there in the family?

- G :

	Past (5 yrs ago)	Present	Future (in 15 years)
Sum of parent's ages	$P-10$	P	$P+30$
Sum of n Children's ages	$C-5n$	C	$C+15n$

- $R: n = ?$
- E, S :
 - $P = 2C$
 - $P - 2C = 0 \rightarrow (1)$
 - $P - 10 = 4(C - 5n)$
 - $P - 4C + 20n = 10 \rightarrow (2)$
 - $P + 30 = C + 15n$
 - $P - C - 15n = -30 \rightarrow (3)$
 - Input SLE in calculator.
- A :
 - $P = 90$
 - $C = 45$
 - $n = 5$

Example

Mary is 24 years old now. Mary is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann now?

- G :

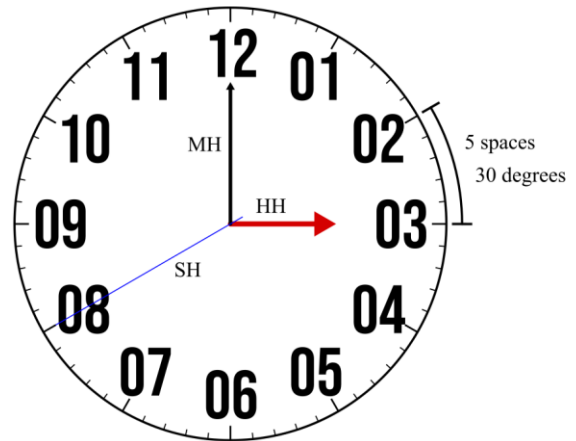
	Past (was)	Present (is)
Mary	$M - x = A$	$M = 24$
Ann	$A - x$	A

- $M = 2(A - x)$
- $R: A = ?$
- E, S :
 - $24 - x = A$
 - $A + x = 24 \rightarrow (1)$

- $2A - 2x = 24 \rightarrow (2)$
- $(2) + 2 \times (1)$
- $2A - 2x + 2A + 2x = 24 + 48$
- $4A = 72$
- $A = 18$

K) Clock Problems

Clock Problem

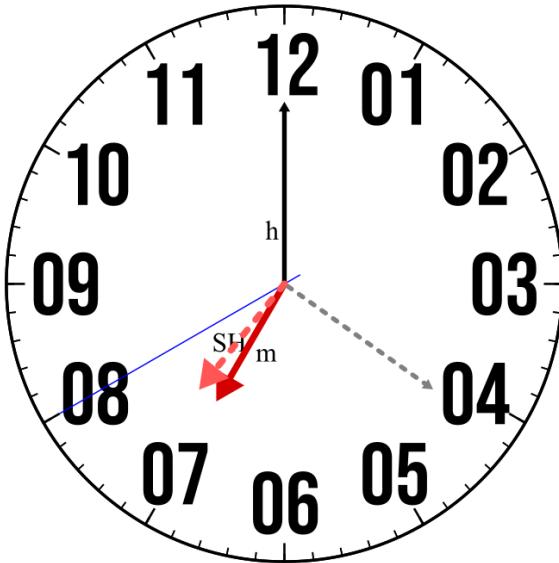


- Ratio of Spaces travelled by hands
 - $SH:MH:HH; 3600:60:5$
 - $\frac{SH}{3600} = \frac{MH}{12} = \frac{HH}{1}$
- $30^\circ = 5 \text{ spaces}$
- Steps for solution:
 - Draw approximate figure.
 - Let x = spaces travelled by fastest hand.
 - Label all spaces.
 - Set-up equations by equating arcs with the same length.
- Steps for my solution (my way)
 - Draw approximate figure.
 - Let
 - h = no. of spaces from 12 mark of hour hand
 - m = no. of spaces from 12 mark of minute hand.
 - s = no. of spaces from 12 mark of second hand.
 - h_0 = value of h when $m = 0$
 - Set-up equations:
 - $h - \frac{m}{12} = h_0$
 - Relationship between h and m .

Example

Find the time after 7 o'clock when the hands (minute hand and hour hand) of the clock are perpendicular for the first time.

- G:



- h is number of spaces of hour mark from 12 mark
- m is number of spaces of minute mark from 12 mark

- R: $t = 7:??$

- E, S:

- $h = 35 + \frac{m}{60}(5)$
- $h - \frac{m}{12} = 35 \rightarrow (1)$
- $h - m = 15 \rightarrow (2)$
- $(1) - (2)$
- $\frac{11}{12}m = 20$
- $m = 21.82 \text{ spaces or minutes}$

- A:

- $t = 7:21.82 \text{ or } 7:21:49.09$

MH and HH Angle Problem Shortcut

- Given angle between MH and HH in final position is known and divisible by 5, x is the number of minutes as pointed by the minute hand.

$$x = (\text{the last hr mark passed by MH}) \left(\frac{60}{11} \right)$$

- Solving previous problem

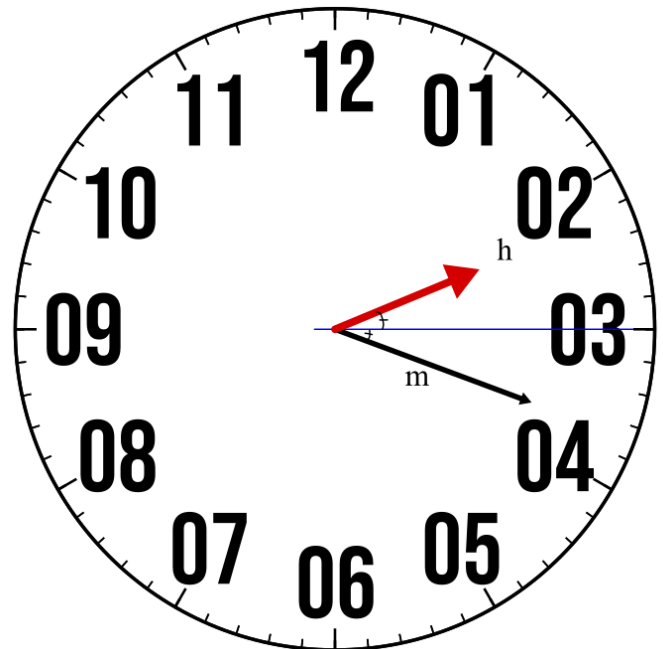
- $x = 4 \left(\frac{60}{11} \right)$
- $x = 21.8181 \text{ min}$

$$t = 7:21:49.09$$

Example

What time after 2 o'clock when the angle between minute-hand and hour-hand is bisected by a line connecting the center and the 3 o'clock mark?

- G:



- h is number of spaces of hour hand from 12 mark
- m is number of spaces of minute hand from 12 mark

- R: $t = 2:??$

- E, S:

- $h = 10 + \frac{m}{60}(5)$
- $h - \frac{m}{12} = 10 \rightarrow (1)$
- $m - 15 = 15 - h$
- $h + m = 30 \rightarrow (2)$
- Input SLE in calculator.
- $h = 11.54$
- $m = 18.46 \text{ spaces or min}$

- A: $t = 2:18.46 \text{ or } 2:18:27.69$

Bisected Problem Shortcut

- If angle between minute-hand and hour-hand is bisected, x is the minute pointed by the minute hand.

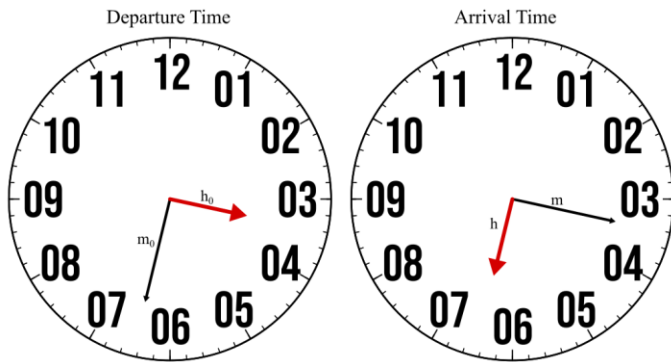
$$x = (\text{hr mark after final position of MH}) \left(\frac{60}{13} \right)$$

- Solving previous problem
 - $x = 4 \left(\frac{60}{13} \right) = 18.46$
 - $t = 2:18.46$

Example

A man left his home past 3 o'clock. After spending between 2 to 3 hours outside, he returned home and noticed that the minute hand and hour hand were exactly interchanged in positions. Find his departure and arrival time.

- G:



- $0 \rightarrow \text{departure}$
 - $\text{no } 0 \rightarrow \text{arrival}$
 - $h \rightarrow$
spaces from 12 mark of hour hand
 - $m \rightarrow$
spaces from 12 mark of minute hand
- R:
 - Arrival Time = ?
 - Departure Time = ?
- E,S:
 - $h = m_0$
 - $m = h_0$
 - $h_0 - \frac{m_0}{12} = 15$
 - $-\frac{h}{12} + m = 15 \rightarrow (1)$
 - $h - \frac{m}{12} = 30 \rightarrow (2)$
 - $h = 31.47$
 - $m = 17.62$

$$m_0 = 31.47$$

- A:
 - Arrival Time is 6:17.62.
 - Departure time is 3:31.47.

Clock Interchange Problem Shortcut

- When clock is interchanged, the number of minutes in the time is
 - $x = (\text{hr mark passed by HH} + \text{hr mark passed by MH} \times 12) \left(\frac{60}{143} \right)$
- Solving previous problem:
 - For departure time:
 - $x = [3 + 6 \times 12] \left(\frac{60}{143} \right)$
 - $x = 31.47 \text{ min.}$
 - Departure time is 3:31.47.
 - For arrival time:
 - $x = [6 + 3 \times 12] \left(\frac{60}{143} \right)$
 - $x = 17.62 \text{ min.}$
 - Arrival time is 6:17.62.

L) Ratio and Proportion Problems

Ratio

- Model
 - $a : b ; c : d$
- a and d are called extremes.
- b and c are called means.
- If $b = c = x$, x is a mean proportional or geometric mean of a and d .
 - $a : x ; x : d$ or $a : x : d$
 - $x = GM = \sqrt{AD}$

Proportion

- Model:
 - $\frac{a}{b} = \frac{c}{d}$
- $a = 1\text{st proportional}$
- $b = 2\text{nd proportional}$
- $c = 3\text{rd proportional}$
- $d = 4\text{th proportional}$

Example

Find the mean proportional to numbers 4 and 25.

- $G: a: x; x: d \rightarrow 4: x; x: 25$
- $R: x = ?$
- $E: x = \sqrt{AD}$
- $S: \text{Substitute.}$
- $A: x = 10$

Example

The ratio of three number, is 2:5:8. Find the largest number if the sum of the numbers is 60.

- $G:$
 - $2: 5: 8; x: y: z$
 - $x + y + z = 60$
- $R: z = ?$
- $E, S:$
 - $\frac{z}{x+y+z} = \text{constant}$
 - $\frac{8}{2+5+8} = \frac{z}{60}$
- $A:$
 - $z = 32$

Example

Find the 3rd proportional to cube of 2 and positive square root of 16.

- $G: 2^3: \sqrt{16}: c$
- $R: c = ?$
- $E, S =$
 - $8: 4: c$
 - $\frac{8}{4} = \frac{4}{c}$
- $A: c = 2$

M) Variation Problems

Proportion or Variation

- $k = \text{constant of proportionality}$
- If Q is directly proportional to x
 - $Q \propto x$
 - $Q = kx$
- If Q is inversely proportional to y
 - $Q \propto \frac{1}{y}$
 - $Q = k \cdot \frac{1}{y}$
- If Q is directly proportional to x and y
 - $Q \propto xy$
 - $Q = kxy$
- If Q is inversely proportional to x and y
 - $Q \propto \frac{1}{xy}$
 - $Q = k \cdot \frac{1}{xy}$
- If Q is directly proportional to x and inversely proportional to y
 - $Q \propto \frac{x}{y}$
 - $Q = k \frac{x}{y}$

Example

The resistance of an electrical conductor is directly proportional to its length and inversely proportional as the square of its diameter. If the length of 600 m and diameter of 10 mm has a resistance of 0.1 Ω , find the length of the same conductor material if it is 20 mm in diameter and resistance is 0.15 Ω .

- $G, R: R \propto \frac{L}{d^2}$
 - $R_1 = 0.1\Omega$
 - $L_1 = 600 \text{ m}$
 - $d_1 = 10 \text{ mm}$
 - $R_2 = 0.15\Omega$
 - $L_2 = ?$
 - $d_2 = 20 \text{ mm}$
- $E, S:$
 - $R = \frac{kL}{d^2}$
 - $k = \frac{Rd^2}{L}$
 - $k = \frac{R_1 d_1^2}{L_1} = \frac{R_2 d_2^2}{L_2}$

$$\circ \frac{0.1(10)^2}{600} = \frac{0.15(20)^2}{L}$$

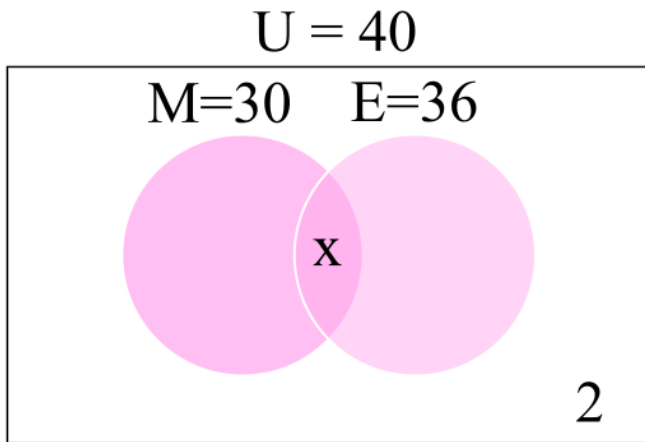
- **A: $x = 3600 m$**

N) Venn Diagram Problems

Example

A class of 40 students took examinations in Math and English. If 30 passed Math, 36 passed English and 2 failed in both subjects, how many students passed in both subjects.

- **G:**

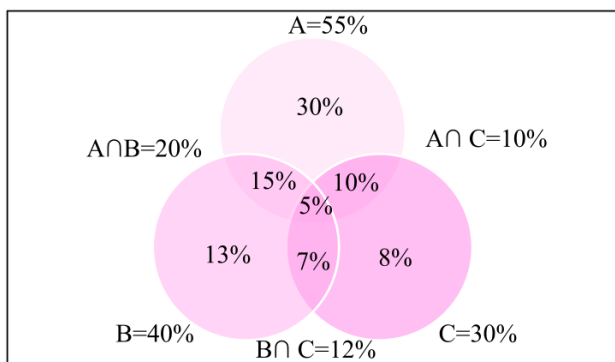


- **R: $x = ?$**
- **E, S:**
 - $30 + 36 - x + 2 = 40$
- **A:**
 - **$x = 28$**

Example

A survey was made on the smokers of 3 cigarette brands A, B, and C. If 55% smoke A, 40% smoke B, 30% smoke C, 20% smoke A and B, 12% smoke B and C, 10% smoke A and C, and 5% smoke A, B, and C. What percent do not smoke.

- **S:**



$$\circ x = 100\% - (30 - 15 - 5 - 10 - 13 - 7 - 8)\%$$

$$\circ x = 12\% \text{ do not smoke.}$$

- Alternative solution:

$$\circ y = A + B + C - (A \cap B) - (A \cap C) - (B \cap C) + (A \cap B \cap C)$$

$$\circ y = 55 + 40 + 30 - 20 - 10 - 7 + 5$$

$$\circ y = 88\% \text{ smokes}$$

$$\circ x = 100\% - 88\% = 12\%$$

$$\circ x = 12\% \text{ do not smoke.}$$

O) Percent Problems

Percent

- $\%x = \frac{x}{R} \cdot 100$
 - $x = \text{part of } R$
 - $R = \text{reference quantity}$
- A is P % more than B.
 - $\%A - B = P\% = \frac{A-B}{B}$
 - $A = (1 + P)B$
- A is P% less than B.
 - $\%A - B = -P\% = \frac{A-B}{B}$
 - $A = (1 - P)B$
- P in the formulas are in decimal form.

Example

A golden retriever dog gained 5.1 lbs. in one month and weighs 65.1 lbs. now. What percent of the weight was gained in one month?

- **G:**
 - $\text{Gain} = 5.1 \text{ lbs}$
 - $W = 65.1 \text{ lbs}$
- **R: $\% \text{Gain} = ?$**
- **E: $\% \text{Gain} = \frac{\text{Gain}}{R}$**
- **S:**
 - $R = W_0 = W - \text{Gain} = 60 \text{ lbs}$
 - Substitute.
- **A: $\% \text{Gain} = 8.5\%$**

Example

A group is made of n engineers and n nurses. If 2 engineers are replaced by 2 nurses, 51% of the group members will be nurses. Find the value of n .

- G :
 - $R = 2n$
 - $\%Nurses = \frac{n+2}{2n} \times 100 = 51\%$
- $R: n = ?$
- E, S : shift-solve.
- **A: $n = 100$**

Example

If Juan is 10% taller than Pedro, and Pedro is 10% taller than Maria, then Juan is taller than Maria by how many percent?

- G :
 - $\%J - P = 10\%$
 - $\%P - M = 10\%$
- $R: \%J - M = ?$
- E, S :
 - $J = 1.1P$
 - $P = 1.1M$
 - $J = 1.1(1.1M)$
 - $J = 1.21M$
 - $\%J - M = \frac{J-M}{M} \times 100$
 - $\%J - M = \frac{1.21M-M}{M} \times 100$
 - $\%J - M = 21\%$
- **A: $\%J - M = 21\%$**

Example

After the price of petroleum oil went up by 10%, a consumer reduced his oil consumption by the same percent. By what percent would his petroleum bill be changed?

- G :
 - $\%P - P_0 = 10\%$
 - $\%C - C_0 = -10\%$
- R :
 - $\%B - B_0 = ?$
- $E: \%B - B_0 = \frac{B-B_0}{B_0} \times 100$
- S :
 - $B = PC$

$$\blacksquare \text{ Pesos} = \text{Peso} / L \times L$$

- $B_0 = P_0 C_0$
- $P = 1.1P_0$
- $C = 0.9C_0$
- $B = 0.99P_0 C_0 = 0.99B_0$
- $\%B - B_0 = \frac{B-B_0}{B_0} \times 100$
- $\%B - B_0 = \frac{0.99B_0-B_0}{B_0} \times 100$
- $\%B - B_0 = -1\%$
- **A: Petroleum will decrease by 1%.**

P) Means of 2 Numbers X and Y

Means of Two Numbers

- Arithmetic Mean
 - $AM = \frac{x+y}{2}$
- Geometric Mean
 - $GM = \sqrt{xy}$
- Harmonic Mean
 - $HM = \frac{2xy}{x+y}$
- Quadratic Mean or Root Mean Square
 - $QM \text{ or } RMS = \sqrt{\frac{x^2+y^2}{2}}$
- Relationship between means:
 - $HM = \frac{GM^2}{AM}$

Example

The arithmetic and geometric mean of two numbers are 10 and 8, respectively. Find their HM.

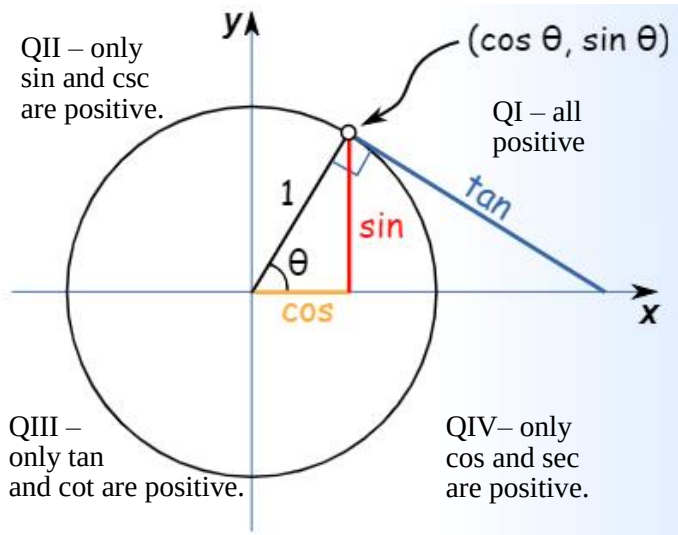
- G :
 - $AM = 10$
 - $GM = 8$
- R :
 - $HM = ?$
- $E: HM = \frac{GM^2}{AM}$
- S : Substitute.
- **A: $HM = 6.4$**

TRIGONOMETRY

trigonometry

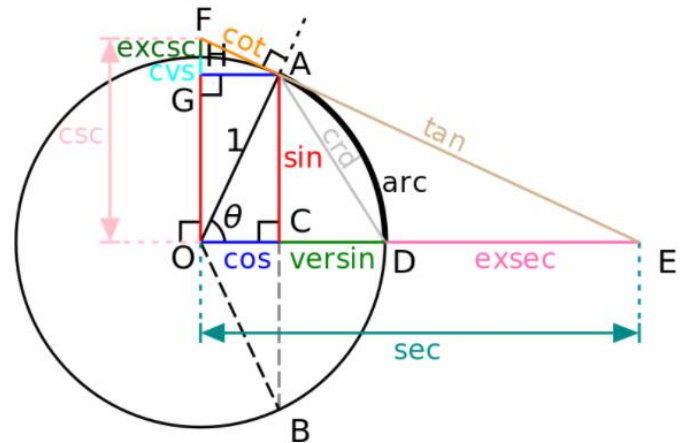
A) Trigonometric and Inverse Trigonometric Functions; Angles and Identities

Six Trigonometric Functions



- Principal Functions (SOH-CAH-TOA)
 - $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$
 - $\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$
 - $\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$
- Reciprocal Functions (CHO-SHA-CAO)
 - $\csc \theta = \frac{1}{\sin \theta} = \frac{\text{hyp}}{\text{opp}} = \frac{r}{y}$
 - $\sec \theta = \frac{1}{\cos \theta} = \frac{\text{hyp}}{\text{adj}} = \frac{r}{x}$
 - $\cot \theta = \frac{1}{\tan \theta} = \frac{\text{adj}}{\text{opp}} = \frac{x}{y}$
- In the unit circle
 - $r = 1$
 - Coordinates: $(x, y) \rightarrow (\cos \theta, \sin \theta)$ are four quadrants QI, QII, QIII, and QIV, as shown.

Special Functions



- Versine:** $\text{versin}(\theta) = 1 - \cos(\theta)$
- Vercosine:** $\text{vercosin}(\theta) = 1 + \cos(\theta)$
- Coversine:** $\text{coversin}(\theta) = 1 - \sin(\theta)$
- Covercosine:** $\text{covercos}(\theta) = 1 + \sin(\theta)$
 - Sine sad dulo indicates negative.
 - Cosine sad ulo indicates positive.
 - Ver sa una indicates cosine.
 - Cover sa duo indicates sine.
- Haversine:** $\text{haversin}(\theta) = \frac{\text{versin}(\theta)}{2}$
- Havercosine:** $\text{havercosin}(\theta) = \frac{\text{vercosin}(\theta)}{2}$
- Hacoversine:** $\text{hacoversin}(\theta) = \frac{\text{coversin}(\theta)}{2}$
- Hacovercosine:** $\text{hacovercos}(\theta) = \frac{\text{covercos}(\theta)}{2}$
- Exsecant:** $\text{exsec}(\theta) = \sec(\theta) - 1$
- Excosecant:** $\text{excsc}(\theta) = \csc(\theta) - 1$

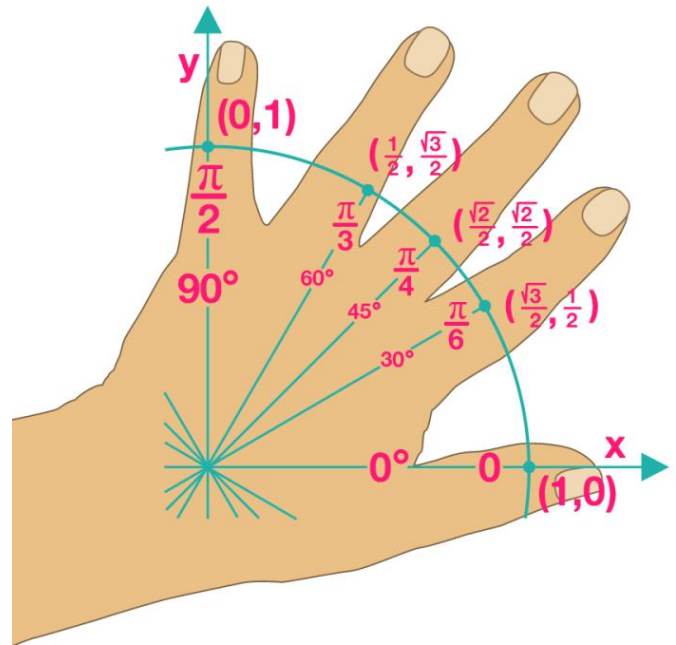
Angle Units Conversion

- 1 rev = 360 degrees
- 1 rev = 2π rad
- 1 rev = 400 grad
- 1 rev = 6400 mils

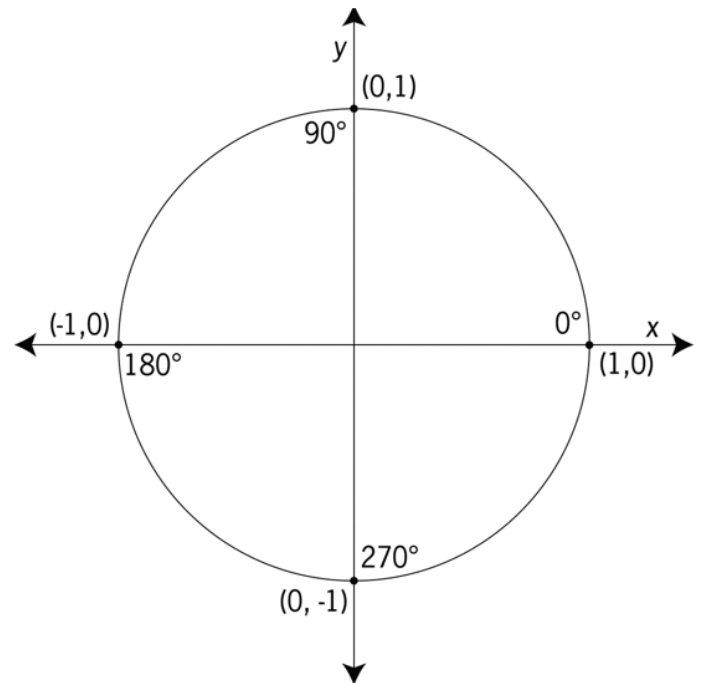
Angle Measurement

- $\theta > 0$ when measured CCW from x-axis
- $\theta < 0$ when measured CW from y-axis

- Complement Angle
 - $\theta_c = 90^\circ - \theta$
 - $\theta + \theta_c = 90^\circ$
 - Forms a right angle.
- Supplement Angle
 - $\theta_s = 180^\circ - \theta$
 - $\theta + \theta_s = 180^\circ$
 - Forms a straight angle.
- Explement Angle
 - $\theta_x = 360^\circ - \theta$
 - $\theta + \theta_x = 360^\circ$
 - Forms one revolution.
- Right Angle – 90 degrees
- Straight Angle – 180 degrees
- Acute Angle
 - $0^\circ < \theta < 90^\circ$
- Obtuse Angle
 - $90^\circ < \theta < 180^\circ$
- Coterminal Angles
 - Angles that when measured CCW from the x-axis terminates at the same direction or line
 - $\theta = \theta + (n)(360^\circ), n = 1, 2, 3, \dots$



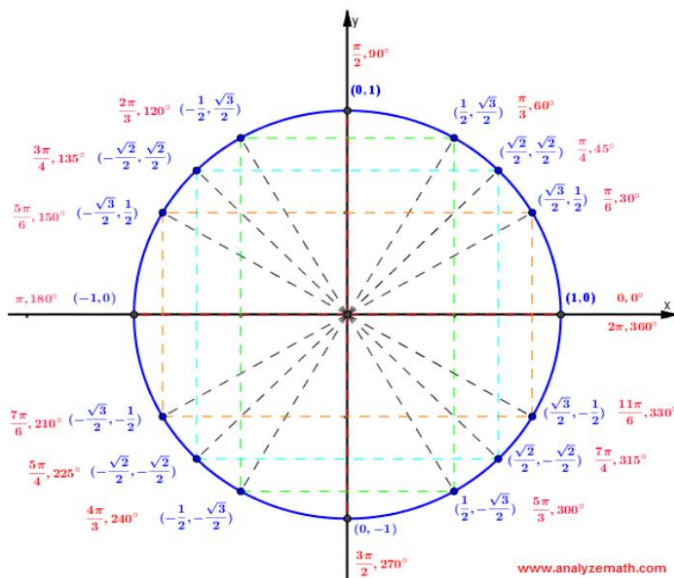
Quadrantal Angles



- Angles exactly on the division of quadrants
- $0^\circ/360^\circ, 90^\circ, 180^\circ, 270^\circ$

Special Angle

- Angle whose cosine and sine can be determined with Pythagorean theorem



Pythagorean Identities

- Memorize first-level identities only. Memorize how to derive the next-level ones.
- $\cos^2 \theta + \sin^2 \theta = 1$**
 - Divide by $\cos^2 \theta$ and $\sin^2 \theta$ to get two more identities.
 - $1 + \tan^2 \theta = \sec^2 \theta$
 - $\cot^2 \theta + 1 = \csc^2 \theta$
- $\sin(A + B) = \sin A \cos B + \cos A \sin B$**
 $\sin(A - B) = \sin A \cos B - \cos A \sin B$
 - Add both equations to get product to sum identity of sine-cosine.
 - $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$**
 - Set $B = A$ in $\sin(A + B)$ to get double angle identity of sine.
 - $\sin(2A) = 2 \sin A \cos A$**
- $\cos(A + B) = \cos A \cos B - \sin A \sin B$**
 $\cos(A - B) = \cos A \cos B + \sin A \sin B$
 - Add both equations to get product to sum identity of cosine-cosine.
 - $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$**
 - Subtract equations to get product to sum identity of sine-sine.
 - $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$**
 - Set $B = A$ in $\cos(A + B)$ to get double angle identity for cosine.
 - $\cos(2A) = \cos^2 A - \sin^2 A$**
 - Set $\cos^2 A = 1 - \sin^2 A$
 - $\cos(2A) = 1 - 2 \sin^2 A$**
 $\sin^2 A = \frac{1}{2}(1 - \cos(2A))$
 - Set $\sin^2 A = 1 - \cos^2 A$
 - $\cos(2A) = 2 \cos^2 A - 1$**
 $\cos^2 A = \frac{1}{2}(1 + \cos(2A))$
- $$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$
 - Set $B = A$ in $\tan(A + B)$
 - $\tan(2A) = \frac{2 \tan A}{1 - \tan^2 A}$
- $\sin(-\theta) = -\sin \theta$**
 $\cos(-\theta) = \cos \theta$
 $\tan(-\theta) = -\tan \theta$
- $\sin \theta = \cos(90^\circ - \theta)$**
 $\cos \theta = \sin(90^\circ \pm \theta)$

Inverse Trigonometric Functions

- In trigonometric functions, the input is the angle, the output is ratio of sides of a right triangle.
- In an inverse trigonometric function, it is the reverse. The input is the ratio, and the output is the angle.
- Inverse Trigonometric Functions
 - $\arcsin x = \sin^{-1} x$
 - $\arccos x = \cos^{-1} x$
 - $\arctan x = \tan^{-1} x$
 - $\operatorname{arccsc} x = \csc^{-1} x$
 - $\operatorname{arcsec} x = \sec^{-1} x$
 - $\operatorname{arccot} x = \cot^{-1} x$
- Note:
 - $\sin^{-1} x \neq \frac{1}{\sin x}$

Example

Evaluate $\sec^{-1} 2$.

- 180°
 - 210°
 - 30°
 - 300°
- $\sec^{-1} 2 = x$
 - $2 = \sec x$
 - $\cos x = \frac{1}{2}$
 - Shift-solve.
 - $x = 60^\circ$, *not in choices*
 - There are no coterminal angles of 60° also such as 420° or -300° .
 - It must be in another quadrant.
 - $\sec x$ is positive where x-coordinate is positive, QI and QIV.
 - 60° is in QI.
 - The other angle in QIV have the same x-coordinate. Mirroring 60 degrees CCW, we get 60 degrees CW or -60° .
 - $x = -60^\circ$, *not in choices*
 - Try to get coterminal angles of -60° . Add or subtract multiples of 360 degrees.
 - $x = -60^\circ + 360^\circ = 300^\circ$
 - Answer is d. 300°**

Example

A certain angle has an explement 5 times the supplement, find the angle.

- $G: \theta_x = 5\theta_s$
- $R: \theta = ?$
- $E, S:$
 - $360^\circ - \theta = 5(180^\circ - \theta)$
 - $360^\circ - \theta = 900^\circ - 5\theta$
 - $4\theta = 540$
 - $\theta = 135^\circ$
- $A: \theta = 135^\circ$

Example

If $\sin A = \frac{4}{5}$ where A is in quadrant II and $\sin B = \frac{7}{25}$ where B is in quadrant I, find $\sin(A + B)$

- $R: \sin(A + B)$
- $S:$
 - $A = \arcsin \frac{4}{5}$
 - Calculator outputs 53.13°
 - This is in quadrant I. Since sine is a function of y, y-coordinate must be the same or the other angle is a mirror of 53.13° about the y-axis.
 - $A = 180^\circ - 53.13^\circ = 126.86^\circ$
 - $B = \arcsin \frac{7}{25}$
 - Calculator outputs 16.26°
 - This is already in quadrant I.
 - $\sin(A + B) = 0.6$

Example

Simplify the equation $(4 \cos y \sin y)(1 - 2 \sin^2 y)$

- a. $\sec 2y$
- b. $\cos 2y$
- c. $\tan 4y$
- d. $\sin 4y$
- $= (2 \sin 2y)(\cos 2y)$
- $= \sin 4y$, Answer is d.
- Alternatively,
 - Set $y =$ some number (test number)
 - Input it in given and all the output and see which will match.

Example

If $\cos x = 2$, find $\cos 3x$.

- $\arccos 2 \rightarrow \text{error in calculator.}$
- x must be complex.
- $x = jy$
- $\cos jy = \cosh y = 2$
- $y = \operatorname{arccosh} 2$
- $x = j \operatorname{arccosh} 2$
- $\cos 3x = \cos(3j \operatorname{arccosh} 2)$
- $= \cosh(3 \operatorname{arccosh} 2)$
- $\cos 3x = 26$

Example

If $\sin 3A = \cos 6B$, then:

- a. $A + B = 180^\circ$
- b. $A + 2B = 30^\circ$
- c. $A - 2B = 30^\circ$
- d. $A + B = 30^\circ$
- $\sin A = \cos(90^\circ - A)$
- $\cos(90^\circ - 3A) = \cos(6B)$
- $90^\circ - 3A = 6B$
- $3A + 6B = 90^\circ$
- $A + 2B = 30^\circ$
- Answer is b.

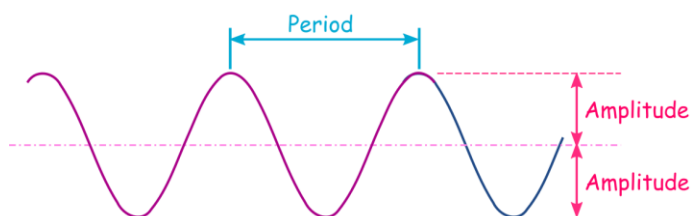
Example

If $77^\circ + 0.4x = \tan^{-1}(\cot(0.25x))$, solve for x .

- a. 10°
- b. 30°
- c. 20°
- d. 40°
- $S:$
 - Shift-solve. Or...
 - Calculate $77^\circ + 0.4x - \tan^{-1}(\cot(0.25x))$ at the different values of x in the choices.
- $A:$
 - c. 20°

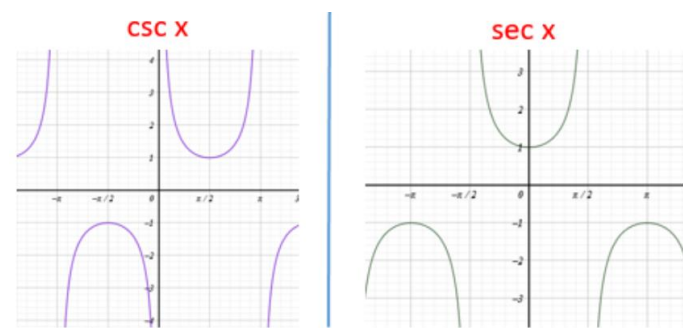
B) Properties of Graph of Trigonometric Functions

Sinusoidal Curve



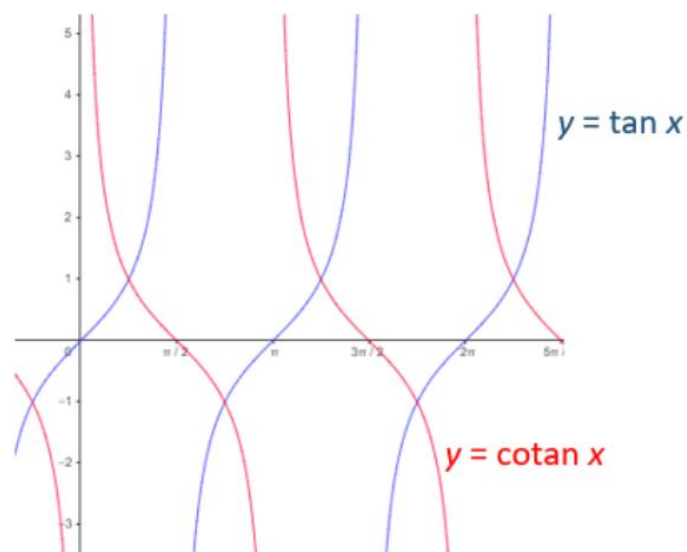
- For pure sine and cosine:
 $y = a \sin bx$ or $y = a \cos bx$
 - Amplitude, A
 - $A = \frac{y_{\max} - y_{\min}}{2}$
 - $A = |a|$
 - Period, T
 - $T = \frac{1}{f}$
 - $T = \frac{2\pi}{|b|}$
 - Frequency, f or ω
 - $\omega = |b| = 2\pi f$
 - $f = \frac{1}{T}$
- For sum of cosine and sine:
 $y = \alpha \cos bx + \beta \sin bx$
 - Amplitude, A
 - $A = \sqrt{\alpha^2 + \beta^2}$

Secant and Cosecant Curve



- $y = a \sec bx$ or $y = a \csc bx$
 - $T = \frac{2\pi}{|b|}$
- Range is $(-\infty, a] \cup [a, +\infty)$

Tangent and Cotangent



- $y = a \tan bx$ or $y = a \cot bx$
 - $T = \frac{\pi}{|b|}$
 - Range is all real numbers.
 - Domain is all real numbers except where the function is undefined.

Example

Find the amplitude and period of the curve represented by $y = 4 \cos 2x$.

- G :
 - $a = 4$
 - $b = 2$
- R :
 - $A = ?$
 - $T = ?$
- E :
 - $A = |a|$
 - $T = \frac{2\pi}{|b|}$
- S :
 - Substitute.
- A :
 - $A = 4$
 - $T = \pi$

Example

Find the period of $y = \tan 3x$.

- G :
 - $a = 1$

- $b = 3$
- R:
 - $T = ?$
- E:
 - $T = \frac{\pi}{|b|}$
- S:
 - Substitute.
- A:
 - $T = \frac{\pi}{3}$

- Imagine that the radius line is rotating. At what quadrant is the x-coordinate ($\cos x$) and y-coordinate ($\sin x$) both increasing.
 - I.e.
 - x-coordinate is going right.
 - y-coordinate is going up.
- This is the case in QIV.
- A: QIV

C) Problems dealing with Right Triangles

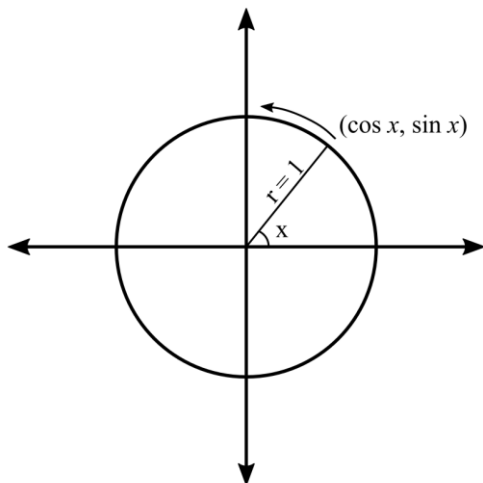
Example

A particle moves in simple harmonic motion in accordance with the equation $s = 3 \sin 2\pi t + 4 \cos 2\pi t$, where s and t are expressed in ft and seconds, respectively. What is the amplitude of its motion?

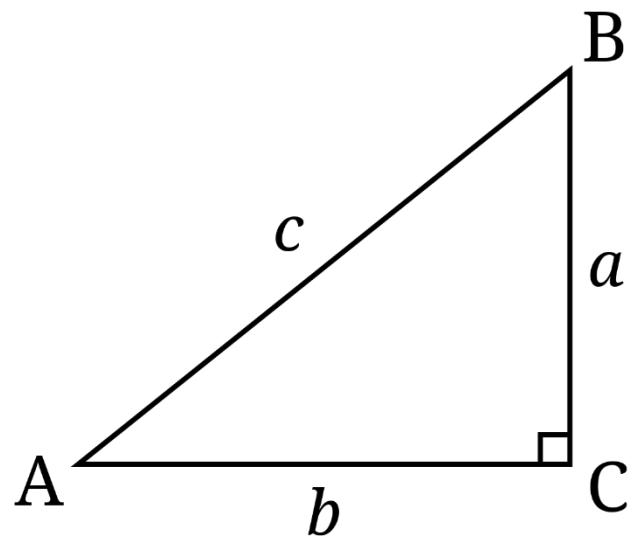
- G:
 - $\alpha = 3$
 - $\beta = 4$
 - $B = 2\pi$
- R:
 - $A = ?$
- E:
 - $A = \sqrt{A^2 + B^2}$
- S:
 - Substitute.
- A:
 - $A = 5 \text{ ft}$

Example

At what quadrant are both $\sin x$ and $\cos x$ increasing?



Right Triangle

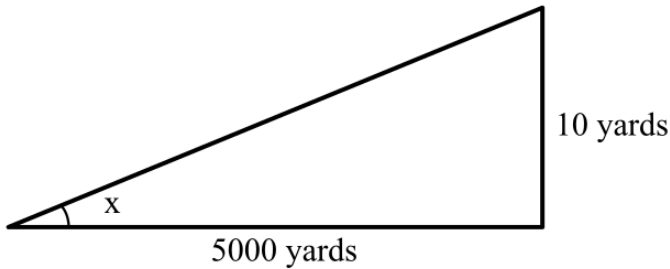


- Pythagorean Theorem
 - $c^2 = a^2 + b^2$
- Sum of Angles
 - $A + B + C = 180^\circ$
 - Since $C = 90^\circ$
 - $A + B = 90^\circ$
- Solutions of Problems with Right Triangles:
 - By Pythagorean Theorem
 - By trigonometric functions

Example

Find the angle in mils subtended by a line 10 yards long at a distance of 5000 yards.

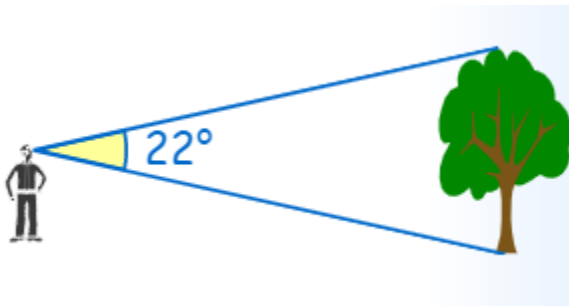
- *G*:



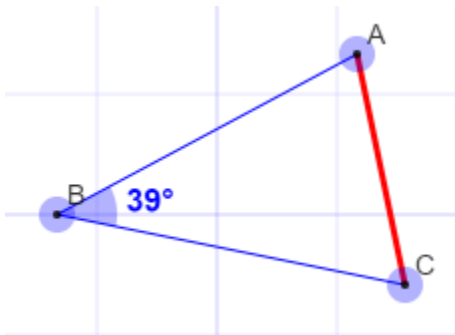
- *R*: $x = ?$
- *S*:
 - $\tan x = \frac{10}{5000}$
 - $x = 0.11^\circ \cdot \frac{6400 \text{ mil}}{360^\circ}$
- *A*: $x = 2.04 \text{ mils}$

To Subtend

- To take up the side opposite an angle or arc
- Ex:



- The angle 22° is subtended by the tree.
- The tree subtends the angle 22° .
- The tree subtends 22° from the eye.

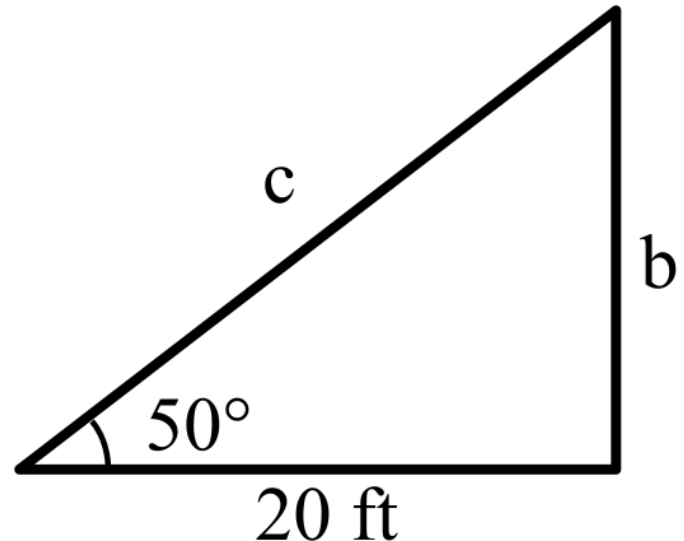


- Side AC subtends angle B.
- Side AC subtends 39° from point B.

Example

A tree broken over by the wind forms a right triangle with the ground. If the broken part makes an angle of 50° with the ground and the top of the tree is now 20 ft from its base, how tall was the tree?

- *G*:

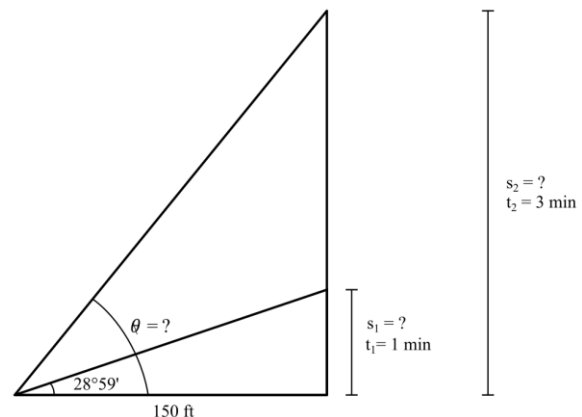


- *R*: $h = c + b = ?$
- *E, S*:
 - $\tan 50^\circ = \frac{b}{20}$
 - $\cos 50^\circ = \frac{20}{c}$
 - $h = c + b$
 - $h = \frac{20}{\cos 50^\circ} + 20 \tan 50^\circ$
- *A*: $h = 54.95 \text{ ft}$

Example

An air balloon flying vertically upward at constant speed is situated 150 m horizontally from an observer. After one minute, it is found that the angle of elevation from the observer is $28^\circ 59'$. What will be the angle of elevation after 3 minutes from its initial position?

- *G*:

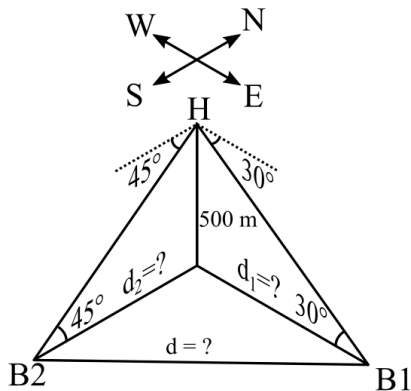


- $R: \theta = ?$
- $E, S:$
 - $\tan 28^\circ 59' = \frac{s_1}{150}$
 - $s_1 = 83.09 \text{ ft}$
 - $v = \frac{s_1}{t_1} = 83.09 \text{ ft/min}$
 - $s_2 = vt_2 = 249.27 \text{ ft}$
 - $\tan \theta = \frac{249.27}{150}$
- $A: \theta = 58.96^\circ \text{ or } 58^\circ 57' 43.45''$

Example

A helicopter is 500m above the sea when the pilot observes two boats, one at the east and the other at the south of him. Find the distance between boats if the angle of depressions is 30° and 45° , respectively.

- $G:$

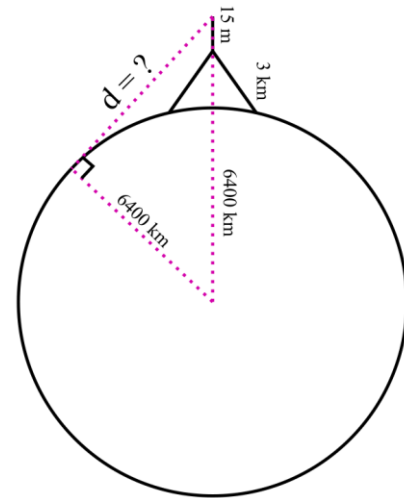


- $R: d = ?$
- $S:$
 - Special Triangles.
 - 45-45 triangles have ratio of sides $1:1:\sqrt{2}$
 - $d_2 = 500 \text{ m}$
 - 30-60 triangles have ratio of sides $2:1:\sqrt{3}$
 - $d_1 = 500\sqrt{3}$
 - $d = \sqrt{d_1^2 + d_2^2}$
- $A: d = 1000\text{m or } 1\text{km}$

Example

A transmitter with a height of 15 m is located on top of a mountain which is 3 km high. What is the furthest distance on the surface of the earth that can be seen from the top of the mountain? Take the radius of the earth to be 6400 km.

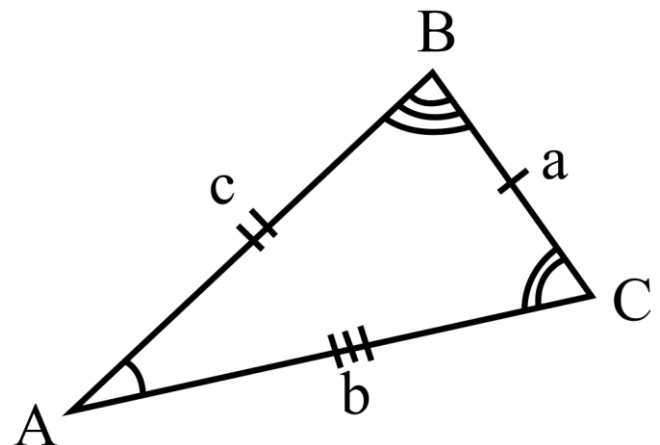
- $G:$



- $R: d = ?$
- $E: a^2 + b^2 = c^2$
- $S: d = \sqrt{(6400 + 3 + 0.015)^2 - (6400)^2}$
- $A: d = 196.47 \text{ km}$

D) Problems Dealing w/ Oblique or Scalene Triangles

Oblique and Scalene Triangles



- They are essentially the same.
- Oblique Triangle
 - A triangle with different angles
- Scalene Triangle
 - A triangle with different sides
- Sine Law:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$
 - If the triangle is inscribed in a circle of radius R.
- Cosine Law:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

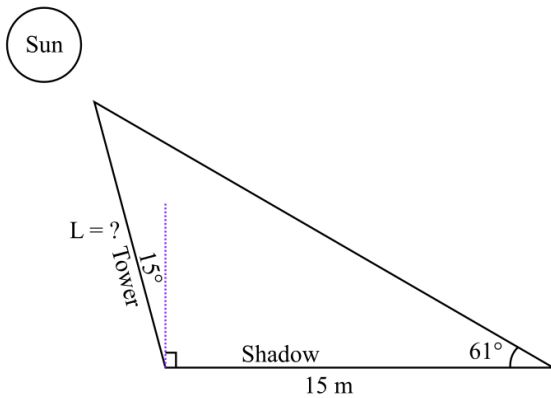
$$c^2 = a^2 + b^2 - 2ab \cos C$$

- Sum of angles is 180°
 - $A + B + C = 180^\circ$
- Opposite of smallest angle is smallest side and vice versa.
- Opposite of biggest angle is biggest side and vice versa.

Example

A tower cast a shadow of 15 m long when the angle of elevation of the sun is 61° . If the tower has leaned 15° from the vertical towards the sun, what is the length of the tower?

- G:

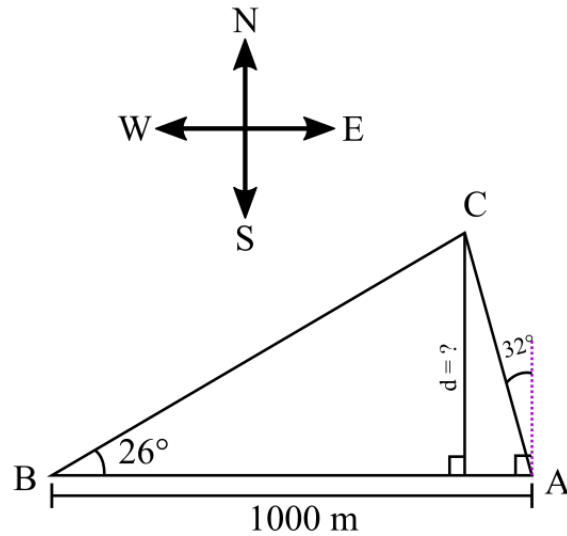


- R: $L = ?$
- E: Sine Law
- S:
 - The angle near sun is $180 - 61 - 90 - 15 = 14^\circ$.
 - $\frac{15}{\sin 14^\circ} = \frac{L}{\sin 61^\circ}$
- A:
 - $L = 54.23 \text{ m}$

Example

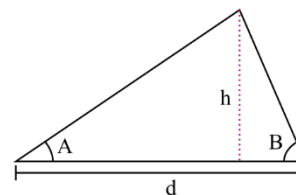
Points A and B, 1000 m apart are plotted on a straight highway running east and west. From A, the bearing of the tower C is 32° W of N and from B, the bearing of C is 26° N of E . Approximate the shortest distance of the tower C to the highway.

- G:



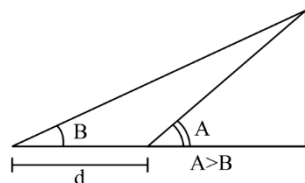
- R: $d = ?$
- E: Sine Law
- S:
 - $A = 90 - 32 = 58^\circ$
 - $C = 180^\circ - A - B = 96^\circ$
 - $\frac{a}{\sin 58^\circ} = \frac{1000}{\sin 96^\circ}$
 - $a = 852.72 \text{ m}$
 - $\sin 26^\circ = \frac{d}{a}$
- A: $d = 373.81 \text{ m}$

Shortcut Solutions



Case I:

$$h = \frac{d \sin A \sin B}{\sin(A + B)}$$



Case II:

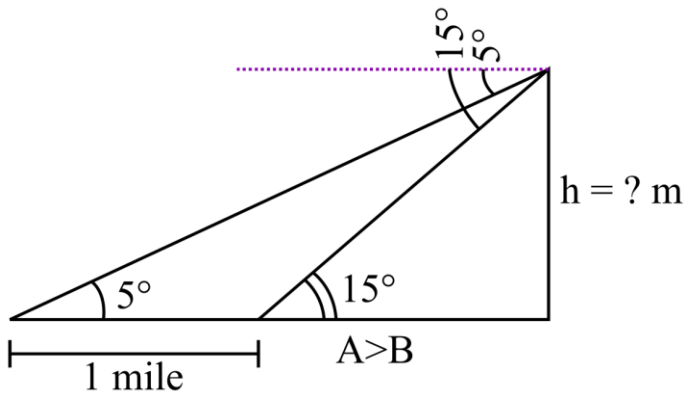
$$h = \frac{d \sin A \sin B}{\sin(A - B)}$$

- Solving previous problem with shortcut:
 - $h = \frac{1000 \sin 26 \sin 58}{\sin(26+58)} = 373.81 \text{ m}$

Example

Two stones 1 mile apart are of the same level as the foot of the hill. The angles of depression of the two stones viewed from the top of the hill are 5° and 15° , respectively. Find the height of the hill in meters.

- *G*:

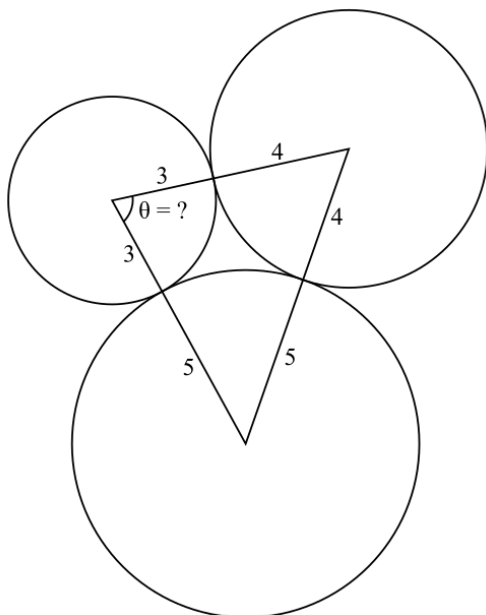


- *R*: $h = ? \text{ m}$
- *E*: $h = \frac{d \sin A \sin B}{\sin(A+B)}$
- *S*: Substitute.
 - $h = 0.130 \text{ mile}$
 - Converting using calculator.
- *A*: $h = 209.06 \text{ km}$

Example

Three circles of radii 3, 4, and 5 inches, respectively are tangent to each other externally. Find the largest angle of the triangle formed by joining the centers.

- *G*:



- *R*: $\theta = ?$

- The angle opposite the longest side is the largest angle.

- *E*: Cosine Law
- *S*: $9^2 = 7^2 + 8^2 - 2(7)(8) \cos \theta$
- *A*: $\theta = 73.40^\circ$

Example

An airplane travels in the direction $N30^\circ W$ at a speed of 500 mph in calm air. The plane encounters a wind with a velocity of 70 mph in the direction $N45^\circ E$. Find the resultant speed and direction of the airplane relative to the ground.

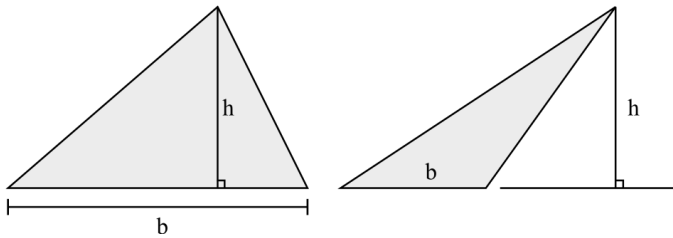
- *G*:
 - $v_a = 500 \text{ mph } N30^\circ W$
 - $v_w = 70 \text{ mph } N45^\circ E$
- *R*:
 - $v_R = v_a + v_w$
- *S*:
 - Convert to phasor.
 - $v_a = 500 \angle (90 + 30)^\circ = 500 \angle 120^\circ$
 - $v_w = 70 \angle (90 - 45)^\circ = 70 \angle 45^\circ$
 - Adding...
 - $525.51 \angle 112.56^\circ$
- *A*: $v_R = 525.51 \text{ mph } N22.56^\circ W$

PLANE MENSURATION

plane mensuration

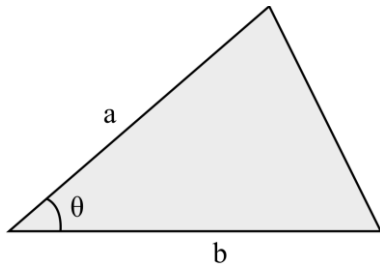
I) Triangles

Basic Formula for Area of Triangle



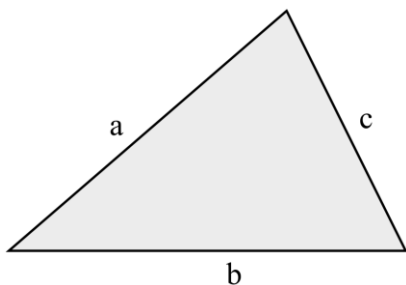
- $A = \frac{1}{2}bh$

Triangle Area Given 2 Sides and Angle Between Them



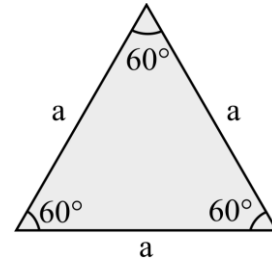
- $A = \frac{1}{2}ab \sin \theta$
- Still $\frac{1}{2}$ base times height
 - The base is b
 - And the height is $a \sin \theta$

Triangle Area Given 3 Sides (Hero(n)'s Formula)



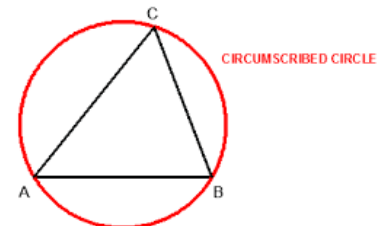
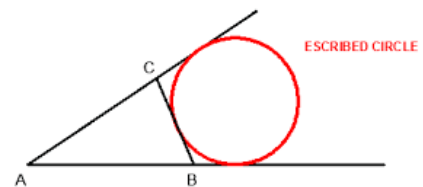
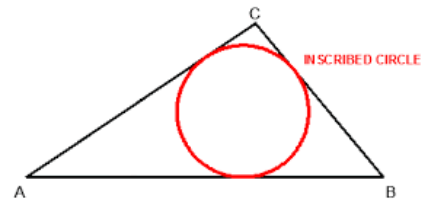
- $A = \sqrt{s(s-a)(s-b)(s-c)}$
 - $s = \frac{1}{2}(a+b+c)$
 - Since $a+b+c$ is the perimeter and s is half the perimeter, s is called the semi-perimeter.

Equilateral Triangle



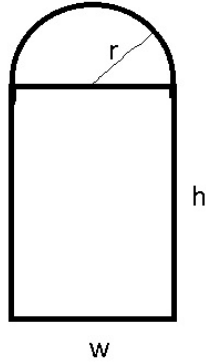
- $A = \frac{1}{2}a^2 \sin 60^\circ$
- $A = \frac{\sqrt{3}}{4}a^2$
 - Since $\sin 60^\circ = \frac{\sqrt{3}}{2}$

Understanding Some Terms



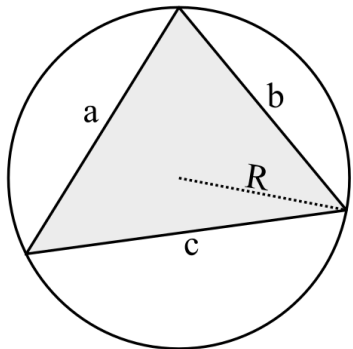
- Inscribed circle – pinasok na circle
- The circle is inscribed in a triangle.
 - Pinasok yung bilog sa triangle.
- Escribed circle – kinalong na circle
- The circle is escribed by a triangle.
 - Yinakap ng triangle ang bilog.
- Circumscribed circle – ipinalibot na circle

- The triangle is circumscribed by a circle.
 - Ang triangle ay pinalibutan ng circle.



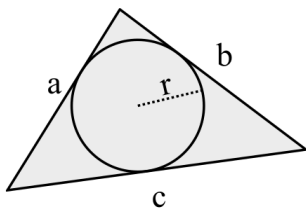
- Surmouted – ipinatong
- Surmounted semi-circle
 - Ipinatong na semi-circle
- The rectangle is surmounted by a semi-circle.
 - Ang rectangle ay pinatungan ng semi-circle.

Area of Inscribed Triangle



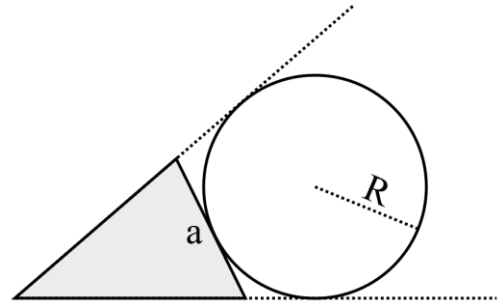
- $A = \frac{abc}{4R}$

Area of Circumscribed Triangle



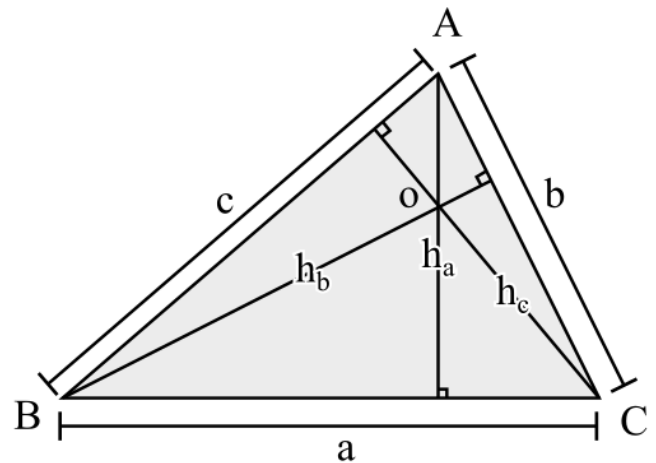
- $A = rs$
 - s is semi-perimeter

Area of Escribed Triangle



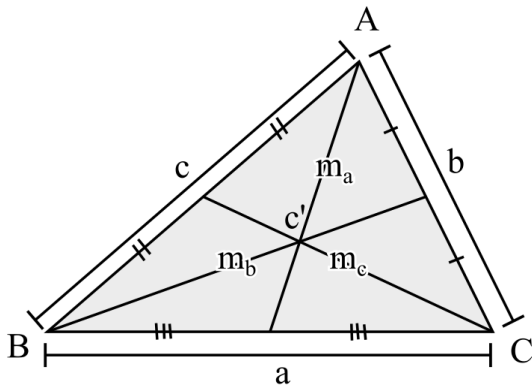
- $A = R(s - a)$
 - s is semi-perimeter

3 Altitudes of a Triangle



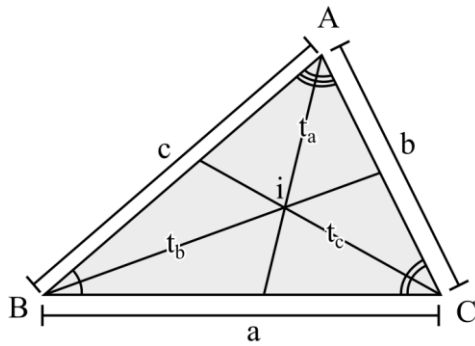
- Altitude or Height
 - Line from vertex to opposite side and is perpendicular to opposite side
- Orthocenter
 - The point of intersections of the altitudes in a triangle
- Formulas for altitude
 - $h_a = \frac{2A}{a}$
 - $h_b = \frac{2A}{b}$
 - $h_c = \frac{2A}{c}$
- Sum of Altitudes
 - $h_a + h_b + h_c = 2A \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$
 - Area A can be substituted with the Heron's formula.

3 Median Lines of a Triangle



- Median Line
 - A line connecting a vertex to the mid-point of opposite side.
- Centroid, c'
 - The point of intersection of median lines
- Formulas for length of median lines
 - $m_a = \frac{1}{2}\sqrt{2(b^2 + c^2) - a^2}$
 - $m_b = \frac{1}{2}\sqrt{2(a^2 + c^2) - b^2}$
 - $m_c = \frac{1}{2}\sqrt{2(a^2 + b^2) - c^2}$

3 Angle Bisectors of a Triangle

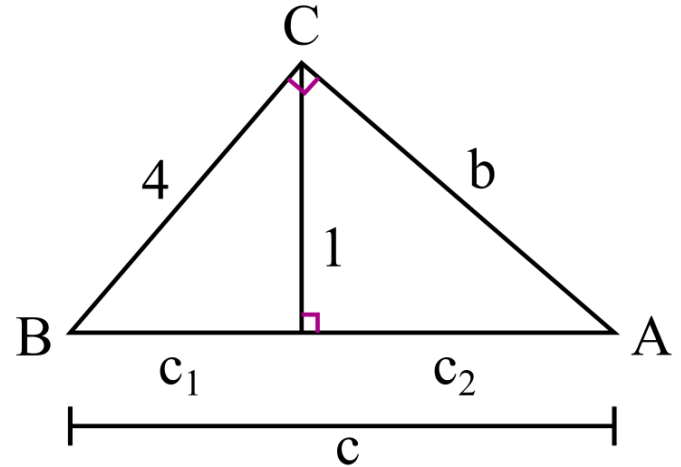


- Angle Bisector
 - A line that divides a vertex angle in half.
- Incenter
 - The center of the inscribed circle
- Formulas for length of angle bisectors:
 - $t_A = \frac{2}{b+c}\sqrt{bcs(s-a)}$
 - $t_B = \frac{2}{a+c}\sqrt{acs(s-b)}$
 - $t_c = \frac{2}{b+c}\sqrt{abs(s-c)}$
- Note:

- When a circle is inscribed in the triangle, the shortest distance from the incenter to a side is the radius of the inscribed circle.

Example

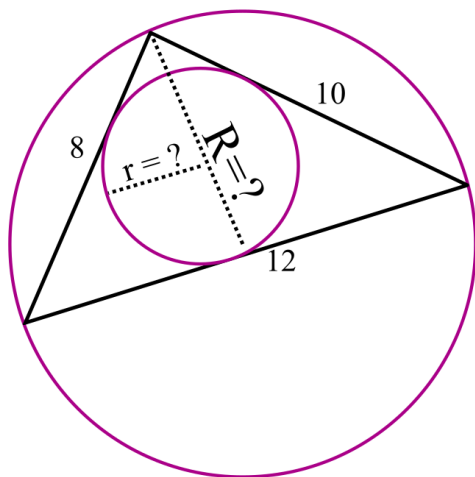
Triangle ABC is a right triangle with right angle at C. If $BC = 4$ and the altitude to the hypotenuse is 1, find the area of the triangle ABC.



- S :
 - $c_1 = \sqrt{4^2 - 1^2} = \sqrt{15}$
 - $c_1 : 1 : c_2$
 - $\frac{c_1}{1} = \frac{1}{c_2}$
 - $c_2 = \frac{1}{c_1} = \frac{1}{\sqrt{15}}$
 - $c = c_1 + c_2$
 - $A = \frac{1}{2}ch_c$
 - $A = \frac{1}{2}\left(\sqrt{15} + \frac{1}{\sqrt{15}}\right)(1)$
 - $A = 2.07 \text{ sq. units}$

Example

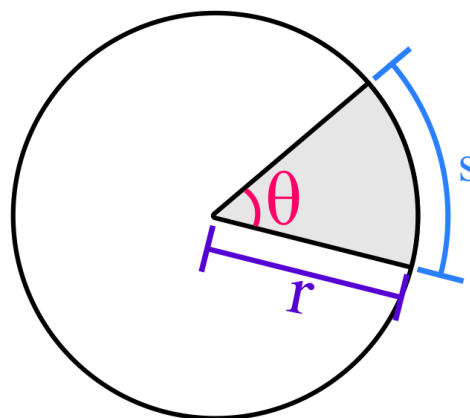
Given is a triangle with sides 8 cm, 10 cm, and 12 cm. Find the radius of the circle that (a) circumscribed the triangle, and (b) is inscribed in the triangle.



- G :
 - $a = 8$
 - $b = 10$
 - $c = 12$
- R :
 - $r = ?$
 - $R = ?$
- E :
 - $A = rs$
 - $A = \frac{abc}{4R}$
 - $A = \sqrt{s(s-a)(s-b)(s-c)}$
 - $s = \frac{1}{2}(a+b+c)$
- S :
 - $s = 15$
 - $A = 39.69 \rightarrow (A)$
 - $r = \frac{(A)}{15} = 2.65$
 - $R = \frac{8 \cdot 10 \cdot 12}{4(A)}$
 - $R = 24.19$
- A :
 - **a. $R = 6.05 \text{ cm}$**
 - **b. $r = 2.65 \text{ cm}$**

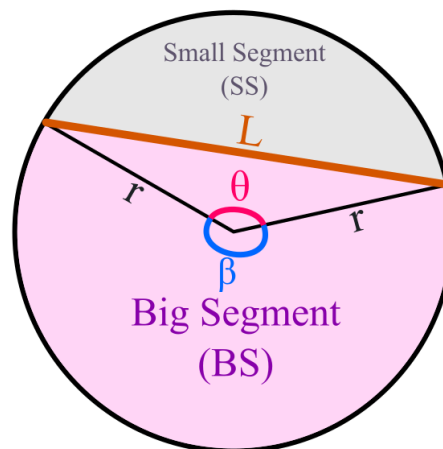
II) Circle and Parts of a Circle

Area of Sector and Circular Arc Length



- $s = r\theta$
 - $s = \text{arclength}$
 - $\theta = \text{angle in radians}$
- $A = \frac{1}{2}rs = \frac{1}{2}r^2\theta$
 - Or: $A = \pi r^2 \cdot \frac{\theta}{2\pi}$

Chord

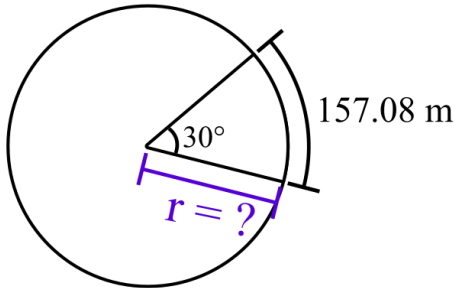


- $L = 2r \sin\left(\frac{\theta}{2}\right)$
 - Derived by dividing the triangle, dividing angle theta. $\sin \frac{\theta}{2} = \frac{L/2}{r}$
- $A_{SS} = \frac{1}{2}r^2(\theta - \sin \theta)$
 - Sector Area – Triangle Area
- $A_{BS} = \frac{1}{2}r^2(\beta - \sin \beta)$
- θ and β are both in radians.

Example

A railroad is to be laid-off in a circular path. What should be the radius if the track is to change direction by 30° at a distance of 157.08 m?

- G:



- $R: r = ?$
- $E: s = r\theta$
- S:
 - 30° in radians is $\frac{\pi}{6}$
 - Substitute.
- **$A: r = 300.00 \text{ m}$**

Example

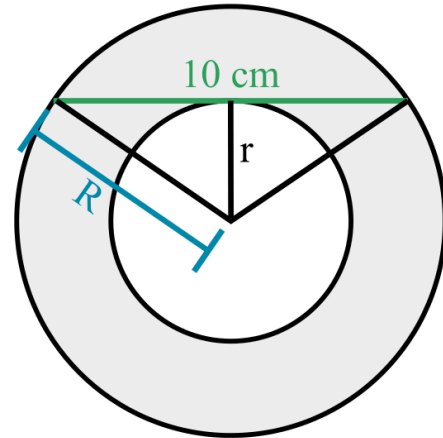
The area of the small segment of a circle with radius 6m is 28 m^2 . Find the length of its cord.

- G:
 - $r = 6 \text{ m}$
 - $A_{SS} = 28 \text{ m}^2$
- $R: L = ?$
- E:
 - $A_{SS} = \frac{1}{2}r^2(\theta - \sin \theta)$
 - $L = 2r \sin \frac{\theta}{2}$
- S:
 - $28 = \frac{1}{2}(6)^2(\theta - \sin \theta)$
 - $\theta = 2.3 \text{ rad}$
 - $L = 2(6) \sin \left(\frac{2.3}{2}\right)$
- A:
 - **$L = 10.96 \text{ m}$**

Example

Given are two concentric circles. If the chord of the big circle is 10 cm and always tangent to the small circle, find the area between the two circles (annular ring)

- G:

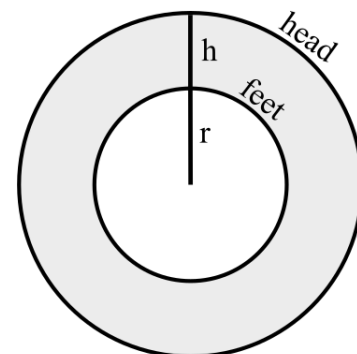


- $R: A = ?$
- S:
 - $R^2 - r^2 = 5^2$
 - $A = \pi(R^2 - r^2)$
 - $A = \pi(5^2)$
 - $A = 25\pi$
- **$A: A = 25\pi \text{ cm}^2$**

Example

Suppose a man 1.7 m tall could walk around the Earth along a great circle, whose radius $r = 6371.64 \text{ km}$. Determine the difference in the distances traveled by his head and feet.

- G:



- $h = 1.7 \text{ m}$
- $r = 6371640 \text{ m}$
- $R: d_{\text{head}} - d_{\text{feet}} = ?$
- S:
 - Distance is the diameter of the circle.

- $d_{head} - d_{feet} = 2\pi(r + h) - 2\pi r$
- $d_{head} - d_{feet} = 2\pi h$
- $d_{head} - d_{feet} = 10.68 \text{ m}$

• **A:** $d_{head} - d_{feet} = 10.68 \text{ m}$

Example

Find the area in cm^2 of the circle circumscribing an isosceles right triangle having an area of 162 cm^2 .

- **G:**
 - $A_{triangle} = 162 \text{ cm}^2$
 - Triangle is isosceles right triangle.
- **R:**
 - $A_{circle} = ?$
- **S:**
 - $A_{triangle} = \frac{1}{2} a^2 = 162$
 - $a = 18$
 - $b = a = 18$
 - $c = 18\sqrt{2}$
 - $A_{triangle} = \frac{abc}{4R}$
 - $R = 12.73$
 - $A_{circle} = \pi R^2$
- **A:** $A_{circle} = 508.94 \text{ cm}^2$
- Alternatively, from c:
 - The hypotenuse c is the diameter of the circle.
 - $R = \frac{c}{2} = \frac{18\sqrt{2}}{2} = 9\sqrt{2}$
 - $A_{circle} = \pi R^2 = 508.94 \text{ cm}^2$

Right triangle Inscribed in a Circle

- If a right triangle is inscribed in a circle,
 - Then the hypotenuse of the right triangle lies on the diameter of the circle.

III) Secant Line and Tangent Line to a Circle

Secant Line

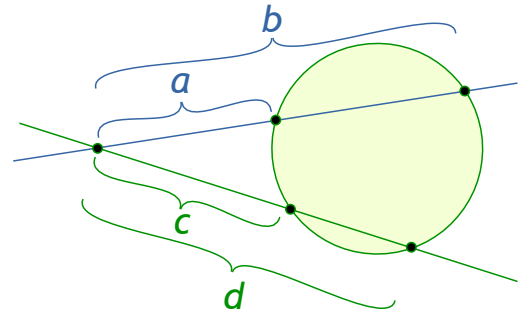
- A line that intersects a curve at two points.

Tangent Line

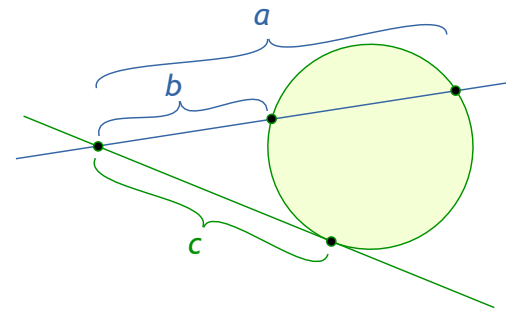
- A line that intersects a curve at only one point

Secant Law

$$a \times b = c \times d$$



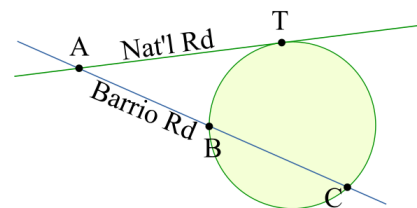
$$a \times b = c^2$$



Example

National road is tangent at point T on the periphery of a circular lake and intersect a barrio road at point A. The barrio road terminates at the edge of the lake at point B. A bridge is to be constructed on the extension of the barrio road to cross the lake. Find the length of the bridge if $AT = 1.5 \text{ km}$ and $AB = 1.25 \text{ km}$.

- **G:**



- $AT = 1.5 \text{ km}$
- $AB = 1.25 \text{ km}$
- **S:**
 - $AB \times AC = AT^2$
 - $AC = 1.8$
 - $L_{bridge} = AC - AB = 0.55 \text{ km}$
- **A:** $L_{bridge} = 550 \text{ m}$

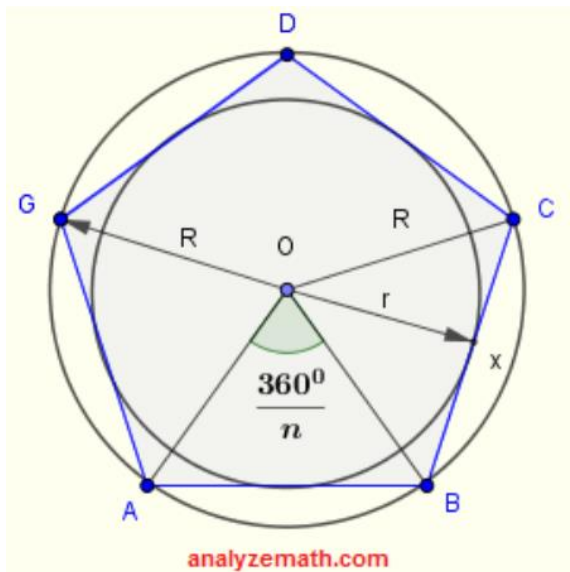
IV) The Regular Polygons

The Regular Polygons

- These are polygons with common (or equal) length of sides.
- Common polygons

Name	No of Sides (n)
Pentagon	5
Hexagon	6
Heptagon	7
Octagon	8
Nonagon	9
Decagon	10
Dodecagon	12
Quindecagon or Pentadecagon	15
Icosagon	20
Hectogon	100
Chiliagon	1000
Myriagon	10000

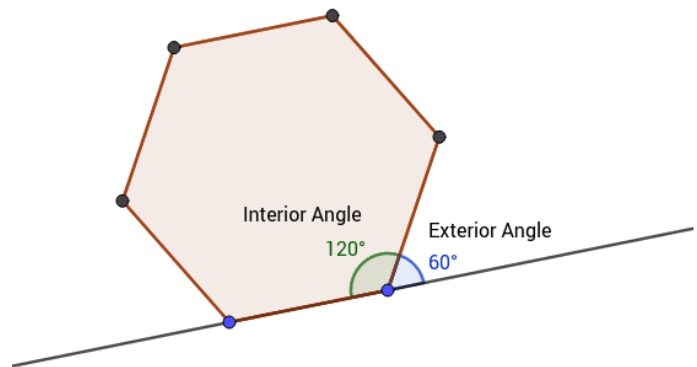
Area Formulas



- Central Angle
 - $\theta = \frac{360^\circ}{n}$
- Area in terms of R:
 - $A = \frac{n}{2} R^2 \sin \theta$

- Area in terms of r
 - $A = nr^2 \tan \left(\frac{\theta}{2} \right)$
 - r is called apothem, the radius of an inscribed circle.
- Area in terms of L (length of sides of a polygon) x in figure
 - $A = \frac{n}{4} L^2 \cot \left(\frac{\theta}{2} \right)$
 - $L = 2R \sin \left(\frac{\theta}{2} \right)$

Exterior and Interior Angles



- I = Interior Angle
- E = Exterior Angle
- Formulas
 - $E = \theta$
 - $\Sigma E = \Sigma \theta = 360^\circ$
 - $\Sigma I = (n - 2)(180^\circ)$
 - $I = \frac{(n-2)(180^\circ)}{n}$

Perimeter

- $P = nL$

No. of Diagonals

- $D = nC2 - n = \frac{n(n-3)}{2}$
 - There are n vertices in an n-polygon.
 - A line can be drawn between two vertices to draw a line. The total number of lines that can be drawn is $nC2$, combination from n vertices choosing two at a time.
 - However, included in these is pair of adjacent vertices forming the sides of the polygon. The sides of the polygon is not a diagonal. The number of sides is n. Thus, n is subtracted.
 - Hence, the formula.

Example

How many sides have a polygon if the sum of its interior angles equals twice the sum of its exterior angles?

- $\Sigma I = 2\Sigma E$
- $(n - 2)(180^\circ) = 2(360^\circ)$
- **$n = 6$ sides**

Example

Given are pentagon and a hexagon with the same area of 12 sq. cm. Find the difference in their perimeters in cm.

- Pentagon:
 - $\theta = \frac{360^\circ}{5} = 72^\circ$
 - $A = \frac{n}{4} L^2 \cot\left(\frac{\theta}{2}\right)$
 - $L_P = 2.6410 \text{ cm}$
- Hexagon:
 - $\theta = \frac{360^\circ}{6} = 60^\circ$
 - $A = \frac{n}{4} L^2 \cot\left(\frac{\theta}{2}\right)$
 - $L_H = 2.1491 \text{ cm}$
- Subtracting Perimeters:
 - $|P_H - P_L| = |6 \times 2.1491 - 5 \times 2.6410|$
 - **$|P_H - P_L| = 0.31 \text{ cm}$**

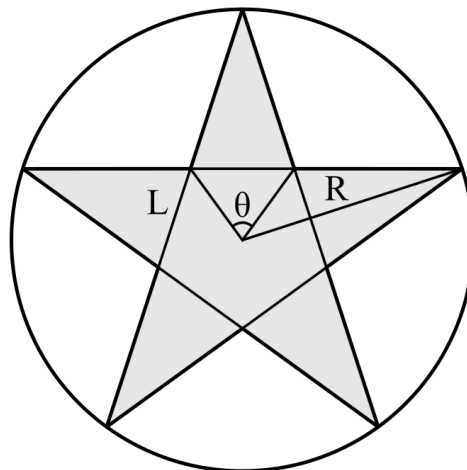
Example

The area of a regular polygon inscribed in a circle is to area of a regular circumscribed polygon having the same number of sides is 3:4. How many sides are there in each polygon.

- $\frac{A_I}{A_C} = \frac{3}{4}$
- $A_I = \frac{n}{2} r^2 \sin \theta$
- $A_C = \frac{n}{2} \left(2r \tan\left(\frac{\theta}{2}\right)\right) r = nr^2 \tan\left(\frac{\theta}{2}\right)$
- $\frac{\frac{n}{2} r^2 \sin \theta}{nr^2 \tan\left(\frac{\theta}{2}\right)} = \frac{3}{4}$
- Cancel n and r^2 . Solve for theta with shift-solve.
- $\theta = 60^\circ$
- $\theta = \frac{360^\circ}{n}$
- $n = 6$
- **Shortcut**
 - $n = \frac{180}{\arccos \sqrt{\frac{A_I}{A_C}}}$

V) Regular N-pointed Star

Regular n-Pointed Star



- Area in terms of R
 - $A = nR^2 \tan\left(\frac{\theta}{2}\right) \cos \theta$
- Area in terms of L
 - $A = \frac{n}{4} L^2 \left(\cot\left(\frac{\theta}{2}\right) + \tan \theta\right)$
- Specific n formulas:
 - For pentagram $n = 5$
 - $A = 1.123 R^2$
 - For octagram $n = 8$
 - $A = 2.343 R^2$

Example

Find the area covered by the pentagram inscribed in a circle with radius 20 cm.

- $A = 1.123 R^2$
- **$A = 449.2 \text{ cm}^2$**

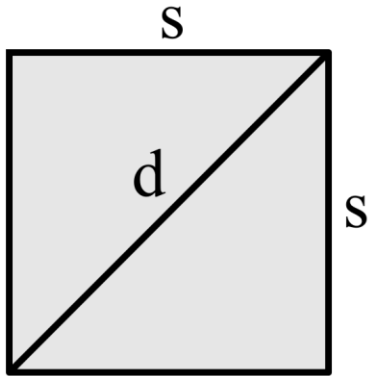
Example

Determine the area of a regular 6-pointed star if the inner regular hexagon has a side of 10 cm.

- $A = \frac{n}{4} L^2 \left(\cot\left(\frac{\theta}{2}\right) + \tan \theta\right)$
- $\theta = \frac{360^\circ}{6} = 60^\circ$
- $L = 10$
- $n = 6$
- Substitute.
- **$A = 519.62 \text{ cm}^2$**

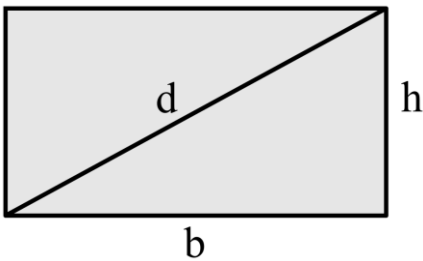
VI) Square and Rectangle

Square



- $A = s^2$
- $P = 4s$
- $d = s\sqrt{2}$

Rectangle



- $A = bh$
- $P = 2b + 2h$
- $d = \sqrt{b^2 + h^2}$

Example

The side of a square is 5 cm less than the side of another square. If the difference in their areas is 55 sq. cm., find the side in cm. of the smaller square.

- G :
 - $s_1 = s_2 - 5$
 - $A_2 - A_1 = 55$
- R :
 - $s_1 = ?$
- E : $A = s^2$
- S :
 - $s_2^2 - s_1^2 = 55$
 - $(s_2 - s_1)(s_2 + s_1) = 55$
 - $s_2 - s_1 = 5$
 - $s_1 + s_2 = 11$

- $2s_1 = 6$
- $s_1 = 3$
- A :
 - $s_1 = 3 \text{ cm}$

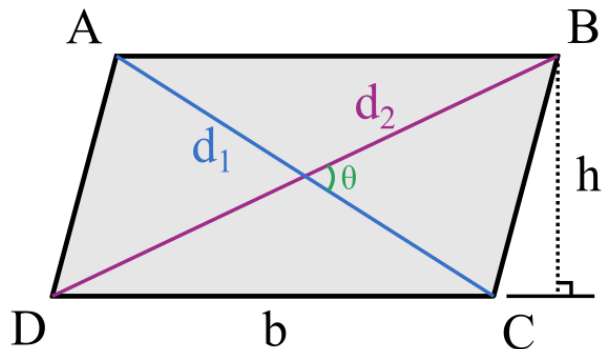
Example

How many acres are contained in a parcel 121 ft wide and 240 yards deep? (1 acre = 43,560 ft²)

- 1 yard = 3 feet
- 240 yards = 720 feet
- $A = 720 \times 121 = 87120 \text{ ft}^2 \times \frac{1 \text{ acre}}{43650 \text{ ft}^2}$
- $A = 2.00 \text{ acre}$

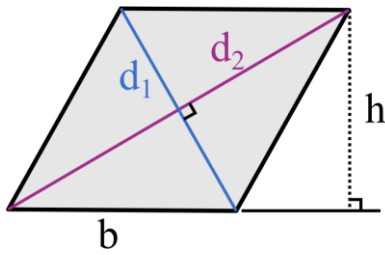
VII) Parallelogram and Rhombus

Parallelogram



- Parallelograms have opposite sides that are parallel.
- Opposite angles are equal.
- $A = bh$
- $A = \frac{1}{2} d_1 d_2 \sin \theta$
 - It doesn't matter whether the other angle between d_1 and d_2 is given.
 - $\sin x = \sin(\pi - x)$

Rhombus



- A parallelogram with equal sides
- Their diagonals are perpendicular to each other.
- The diagonals bisect each other.
- $A = bh$
- $A = \frac{1}{2}d_1d_2$

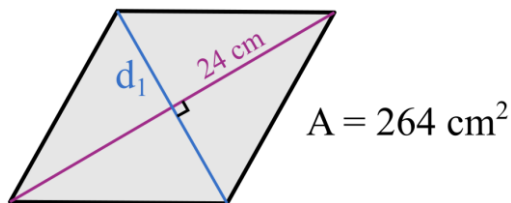
Example

Given is a parallelogram with angle $A = 5x + 2$ and angle $C = 6x - 4$. Find angle A .

- A and C are opposite angles.
- Angle A and C are equal.
- $5x + 2 = 6x - 4$
- $x = 6$
- $A = 5(6) + 2$
- $A = 32^\circ$

Example

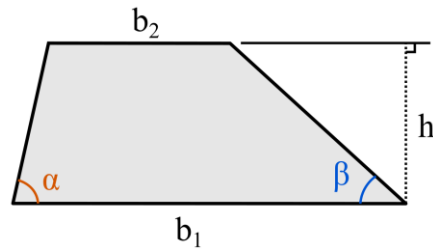
The area of the rhombus is 264 sq. cm. If one of the diagonals is 24 cm. long, find the length of the other diagonal.



- $A = \frac{1}{2}d_1d_2$
- $264 = \frac{1}{2}d_1(24)$
- $d_1 = 22 \text{ cm}$

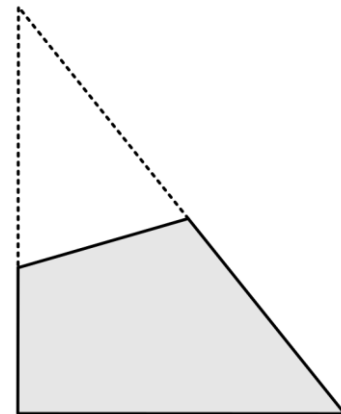
VIII) Trapezoid and Trapezium

Trapezoid



- Quadrilateral with a pair of sides parallel.
- $A = \left(\frac{b_1+b_2}{2}\right)h$
- $A = \frac{b_1^2-b_2^2}{2(\cot\alpha+\cot\beta)}$

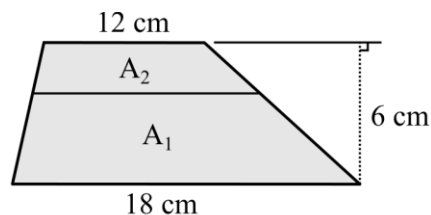
Trapezium



- Quadrilateral with no pair of parallel sides.
- Area is area of big triangle minus area of small triangle.

Example

A trapezoid 6 cm in altitude have bases of 12 cm and 18 cm, respectively. If it is divided into two parts by a line parallel to the bases such that the ratio of the areas of the two parts formed is 2:3, compute the length of the dividing line in cm.

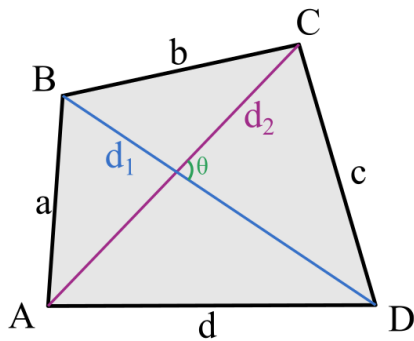


- Let L = length of dividing line
- $\sqrt{\frac{A_2}{A_1}} = \frac{2}{3} = \frac{L}{18} = \frac{12}{L}$
- $L = 14.70 \text{ cm}$

- Whole solution:
 - Draw a line from top right point downwards parallel to left side.
 - This makes a parallelogram at the left and two similar triangles on the right.
 - Proceed with this knowledge.
 - Too lengthy though. Not worth it. Char.

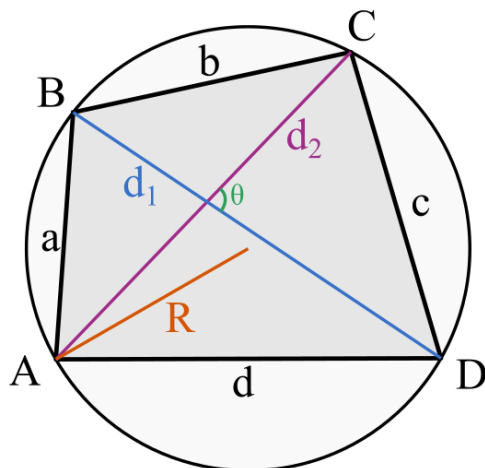
IX) Quadrilateral

Quadrilateral



- $A = \frac{1}{2} d_1 d_2 \sin \theta$
- $A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \phi}$
 - $s = \frac{1}{2}(a+b+c+d)$ is the semi-perimeter
 - $\phi = \frac{A+C}{2} = \frac{B+D}{2}$
 - Called Brahmagupta's Formula

Cyclic Quadrilateral

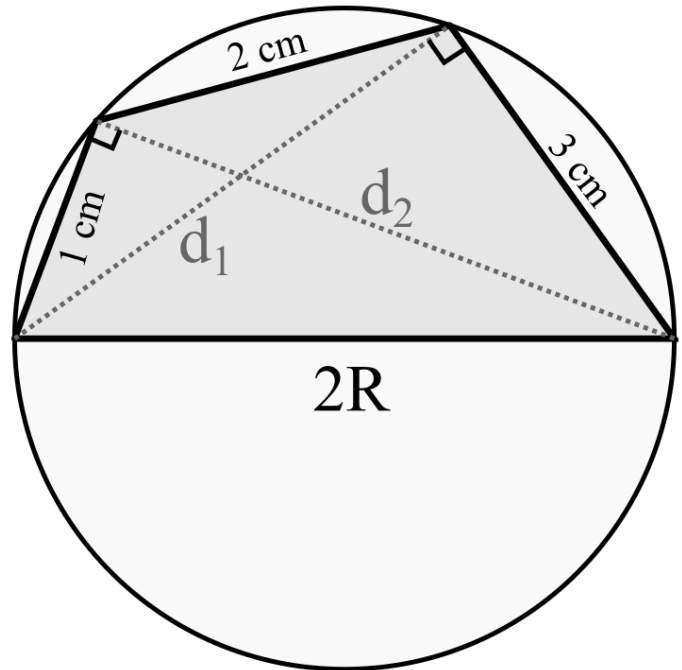


- Quadrilateral inscribed in a circle or circumscribed by a circle
- Formula or Properties
 - $A + C = B + D$

- $ac + bd = d_1 d_2$
 - Ptolemy's Theorem
- $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
- $R = \frac{\sqrt{(ab+cd)(ad+bc)(ac+bd)}}{4A}$

Example

A quadrilateral is inscribed in a circle in such that one of the sides is the diameter of the circle. If the three other sides are 1 cm, 2cm, and 3cm respectively, find the area enclosed by quadrilateral.



- $a = 1$
- $b = 2$
- $c = 3$
- $d = 2R$
- $ac + bd = d_1 d_2$
- $3 + 2d = d_1 d_2$
- Triangles inscribed in a semicircle are right triangles.
- $d_1 = \sqrt{d^2 - 3^2}$
- $d_2 = \sqrt{d^2 - 1^2}$
- $3 + 2d = \sqrt{d^2 - 9} \sqrt{d^2 - 1}$
- $d = 4.11 \rightarrow (D)$
- $s = \frac{1}{2}(1 + 2 + 3 + (D))$
- $s = 5.06 \rightarrow (A)$
- $A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$
- $A = 4.90 \text{ cm}^2$

SOLID MENSURATION

solid mensuration

Solid Mensuration

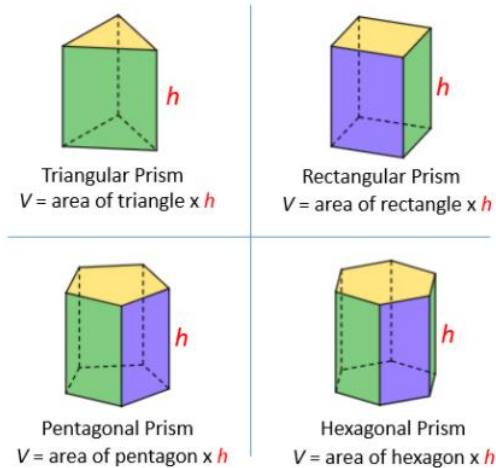
- Deals with surface area and volume computation

1) Prism

Prism

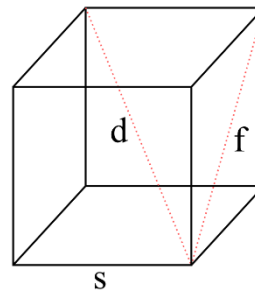
- A solid with constant-cross-sectional area and with parallel end areas
- Named after its base, e.g., square prisms

Prism Volume and Surface Area



- $V = Bh = A_B h$
 - Volume = base area x height
- $S_L = P_B h$
 - Lateral Surface Area = base perimeter x height
 - Lateral Surface Area is the surface area of the sides excluding the top and bottom base.
- $S_T = P_B h + 2A_B$
 - Total Surface Area = lateral surface area + 2 times base area

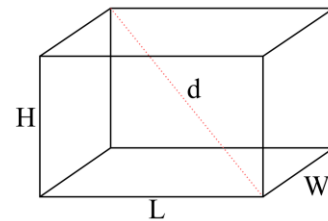
Special Case 1: Cube or Hexahedron



s = side length
 f = face diagonal length
 d = diagonal length

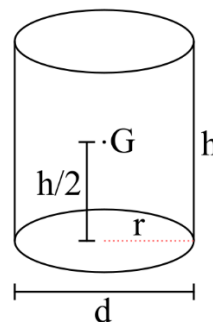
- $V = s^3$
- $S_T = 6s^2$
- $s^2 + f^2 = d^2$
- $f = s\sqrt{2}$
- $d = s\sqrt{3}$

Special Case 2: Rectangular Parallelepiped



- $V = LWH$
- $S_T = 2(LW + LH + WH)$
- $d = \sqrt{L^2 + W^2 + H^2}$

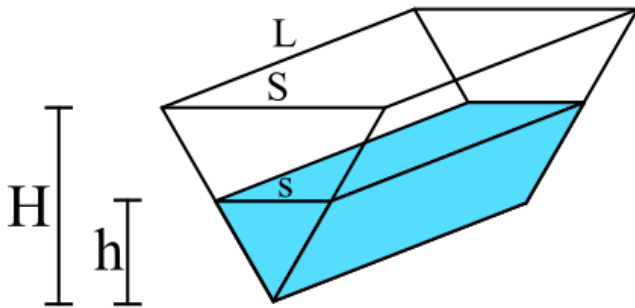
Special Case 3: Right Circular Cylinder



- $V = \pi r^2 h = \frac{\pi}{4} d^2 h$
- $S_L = 2\pi r h = \pi d h$
- $S_T = 2\pi r h + 2\pi r^2$
- $A_B = \pi r^2 = \frac{\pi}{4} d^2$
- $P_B = 2\pi r = \pi d$

Example

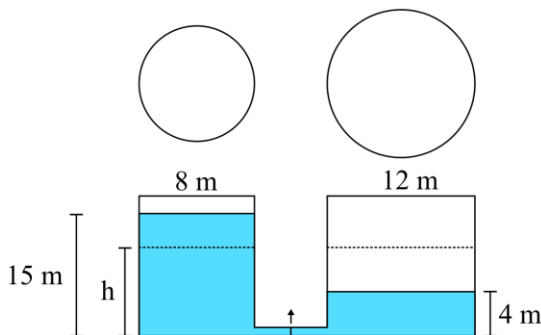
A trough having an equilateral triangle end sections has side equal to 0.3 m and 6 m long. Find the volume of water in the trough in the trough if its depth is $\frac{1}{2}$ the depth of the trough.



- Similar triangles...
- $\frac{h}{H} = \frac{s}{S}$
- $s = S \frac{h}{H}$
- $s = 0.3 \frac{\frac{1}{2}H}{H}$
- $s = 0.15$
- $V_{water} = \frac{1}{2} (0.15)^2 (6) \sin 60^\circ$
- **$S = 0.0585 \text{ m}^3$**

Example

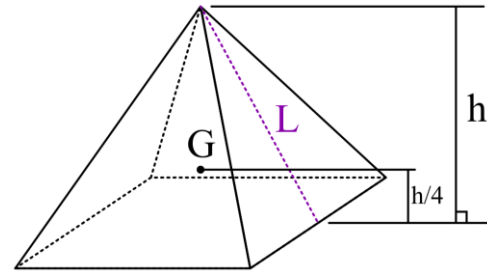
Two vertical cylindrical tanks with diameters of 8 m and 12 m respectively are joined by a pipe at their bases. If the tanks are filled with water to a depth of 15 m and 4 m respectively, find the final depth of the water in the tanks when the valve in the pipe is opened.



- Volume Before = Volume After
- $(\pi r^2 h)_{\text{before}} = (\pi r^2 h)_{\text{after}}$
- $\pi(4)^2(15) + \pi(6)^2(4) = \pi(4)^2 h + \pi(6)^2 h$
- **$h = 7.38 \text{ m}$**

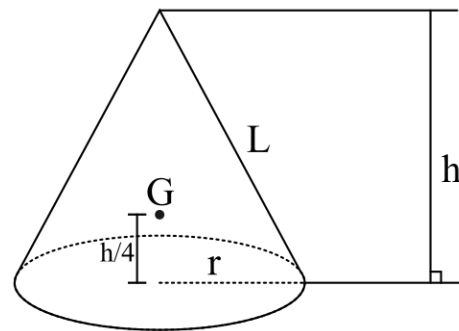
II) Pyramid and Cone

Pyramid



- L = Slant Height
- Point G = Centroid
- $V = \frac{1}{3} A_B h$
- $S_L = \frac{1}{2} P_B L$

Right Circular Cone



- $V = \frac{1}{3} \pi r^2 h$
- $S_L = \pi r L$
- $L = \sqrt{r^2 + h^2}$

Example

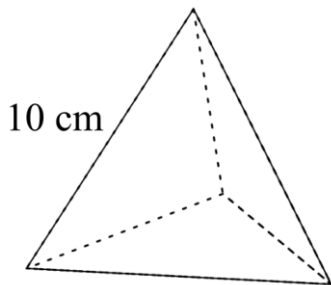
A conical vessel has a height of 24 cm and a base diameter of 12 cm. Find (a) the amount of material used, and (b) the volume of water in the vessel if it is 18 cm deep above its vertex.

- $H = 24 \text{ cm}$
- $D = 12 \text{ cm} \rightarrow R = 6 \text{ cm}$
- $h = 18 \text{ cm}$
- (a)
 - $S_L = \pi R \sqrt{R^2 + H^2}$
 - $S_L = \pi(6) \sqrt{6^2 + 24^2}$
 - **$S_L = 466.31 \text{ cm}^2$**
- (b)
 - Similar triangles...

- $\frac{r}{R} = \frac{h}{H}$
- $\frac{r}{6} = \frac{18}{24}$
- $r = 4.5$
- $V = \frac{1}{3}\pi r^2 h$
- $V = \frac{1}{3}\pi(4.5)^2(18)$
- $V = 381.70 \text{ cm}^3$

Example

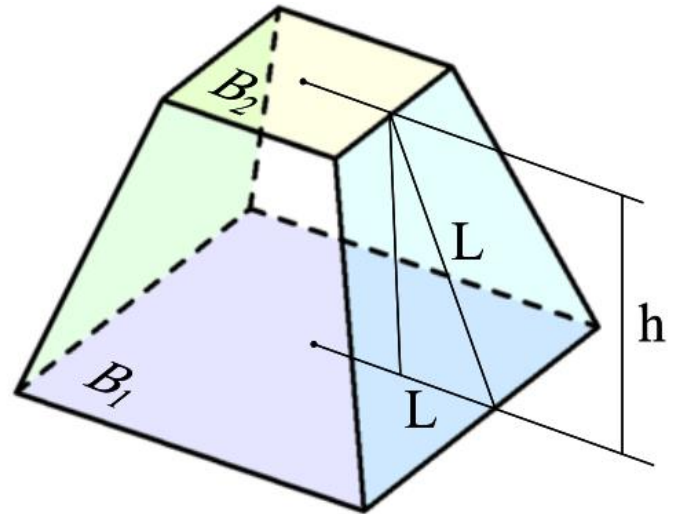
Find the volume and total surface area of a tetrahedron with edge 10 cm long.



- $s = 10 \text{ cm}$
- A tetrahedron has four faces that are equilateral triangles.
- Volume
 - $h = \sqrt{\frac{2}{3}}s$
 - $V = \frac{1}{3}A_b h$
 - $V = \frac{1}{3}\left(\frac{1}{2}s^2 \sin 60^\circ\right)h$
 - $V = \frac{1}{3}\left(\frac{1}{2}(10)^2 \sin 60^\circ\right)\sqrt{\frac{2}{3}}(10)$
 - $V = 117.85 \text{ cm}^3$
- Area
 - $S_T = 4A_{\text{face}}$
 - $A_{\text{face}} = \frac{1}{2}s^2 \sin 60^\circ$
 - $S_T = 2s^2 \sin 60^\circ$
 - $S_T = 173.21 \text{ cm}^2$

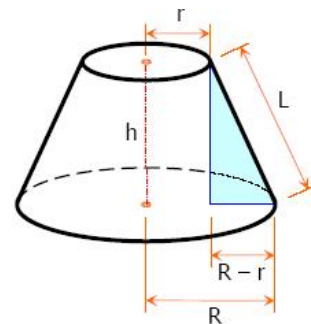
III) Frustum of A Pyramid and Frustum of a Cone

Frustum of a Pyramid



- $V = \frac{1}{3}(A_{B1} + \sqrt{A_{B1}A_{B2}} + A_{B2})h$
- $S_L = \frac{P_{B1} + P_{B2}}{2} \times L$

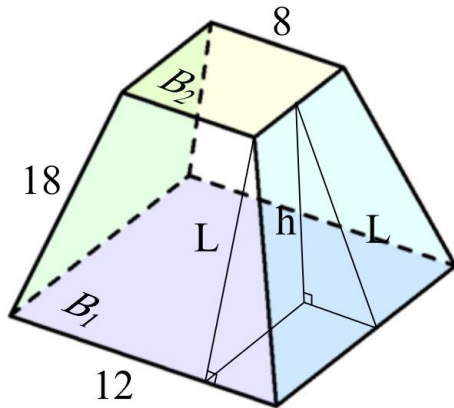
Frustum of a Cone



- $V = \frac{1}{3}\pi(R^2 + Rr + r^2)h$
- $S_L = \pi(R + r)L$
- $L = \sqrt{(R - r)^2 + h^2}$

Example

A frustum of a regular pyramid has a lower base of 12 cm x 12 cm and an upper base of 8 cm x 8 cm. It has a lateral edge of 18 cm. Find its (a) lateral surface area, (b) volume.



- $S = 12 \text{ cm}$
- $s = 8 \text{ cm}$
- $a = 18 \text{ cm}$
- $L = \sqrt{18^2 - \left(\frac{12-8}{2}\right)^2} = 17.89 \rightarrow (x)$
- (a) Lateral Surface Area
 - $S_L = \frac{P_{B1} + P_{B2}}{2} \times L$
 - $S_L = \frac{4S + 4s}{2} \times L$
 - $S_L = \frac{4 \times 12 + 4 \times 8}{2} \times (x)$
 - $S_L = 715.54 \text{ cm}$
- (b) Volume
 - $h = \sqrt{L^2 - \left(\frac{S-s}{2}\right)^2}$
 - $h = \sqrt{(x)^2 - \left(\frac{12-8}{2}\right)^2} = 17.78 \rightarrow (y)$
 - $V = \frac{1}{3}(A_{B1} + \sqrt{A_{B1}A_{B2}} + A_{B2})h$
 - $V = \frac{1}{3}(s^2 + sS + S^2)h$
 - $V = \frac{1}{3}(8^2 + 8 \times 12 + 12^2)(y)$
 - $V = 1801.34 \text{ cm}^3$

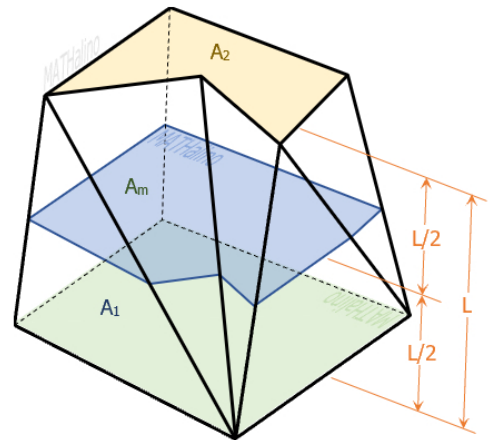
Example

Find the volume and lateral surface area of the frustum of a cone with base radii of 3 cm and 6 cm if it is 10 cm tall.

- $r = 3 \text{ cm}$
- $R = 6 \text{ cm}$
- $h = 10 \text{ cm}$
- Volume
 - $V = \frac{1}{3}\pi(r^2 + rR + R^2)h$
 - $V = \frac{1}{3}\pi(3^2 + 3 \cdot 6 + 6^2)(10)$
 - $V = 659.73 \text{ cm}^3$
- Area
 - $L = \sqrt{(R-r)^2 + h^2}$
 - $L = \sqrt{(6-3)^2 + 10^2} \rightarrow (x)$
 - $S_L = \pi(R+r)L$
 - $S_L = \pi(6+3)(x)$
 - $S_L = 295.19 \text{ cm}^2$

IV) The Prismoidal Formula

Prismatoid



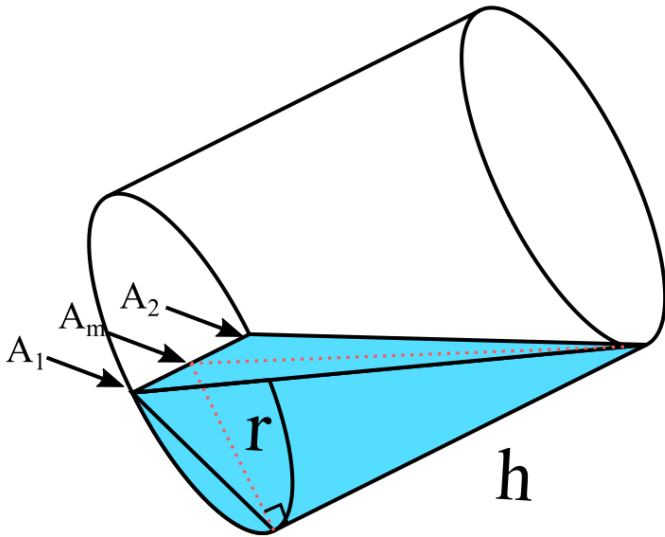
- Polyhedron having bases containing all its vertices in two parallel planes.
- The lateral faces are triangles or trapezoids or parallelograms with one side lying on one of its two bases.
- $V = \frac{L}{6}(A_1 + 4A_m + A_2)$

Prismoidal Formula

- If the bases are consecutively denoted as 1, 2, 3, consecutively.
- The volume is:
 - $V = \frac{D}{3}(A_1 + 4\Sigma A_{even} + 2\Sigma A_{odd} + A_n)$
 - n number of parallel areas
 - A_n is last base area.
 - A_1 is first base area.
 - ΣA_{even} is the sum of even ordered areas.
 - ΣA_{odd} is the sum of odd ordered areas.
 - D is common distance between areas
 - $D = \frac{L}{n-1}$

Example

A cylindrical tank 1.5 m tall and 0.75 m in diameter is full of water. If the tank is slowly tilted until the water surface coincide with the diameter of the base, find the volume of water left in the tank.



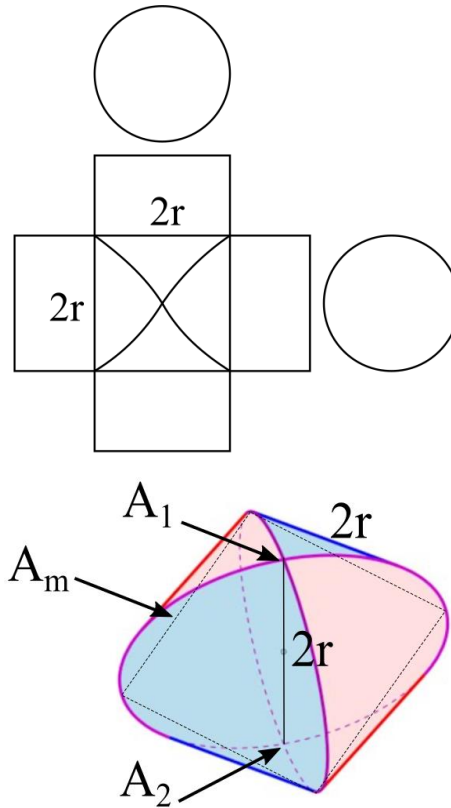
- $r = \frac{0.75 \text{ m}}{2} = 0.375 \text{ m}$
- $h = 1.5 \text{ m}$
- $V = ?$
- $A_1 = A_2 = 0$
- $A_m = \frac{1}{2}r^2h$
- $L = 2r$
- $V = \frac{L}{6}(A_1 + 4A_m + A_2)$

- $V = \frac{2r}{6}(0 + 4(\frac{1}{2}rh) + 0)$
- $V = \frac{2}{3}r^2h$
- $V = \frac{2}{3}(0.375)(1.5)$
- $V = 0.1406 \text{ m}^3$

Volume of water takes a shape of a wedge of a cylinder.

Example

Two cylinders with radii 20 cm each are made to cross each other. Find the volume of the solid common between the cylinders.

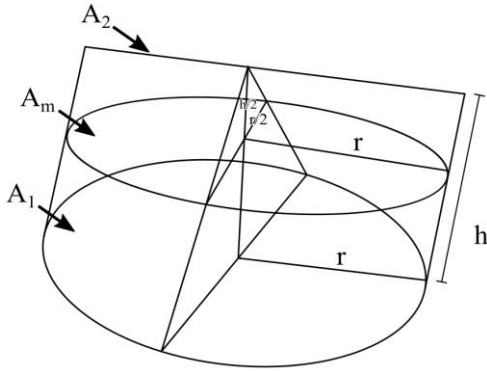


- $r = 20 \text{ cm}$
- $V = ?$
- $A_1 = A_2 = 0$
- $A_m = (2r)^2$
- $A_m = 4r^2$
- $L = 2r$
- $V = \frac{L}{6}(A_1 + 4A_m + A_2)$
- $V = \frac{2r}{6}(0 + 4(4r^2) + 0)$
- $V = \frac{16}{3}r^3; r = 20 \text{ cm}$
- $V = 42666.67 \text{ cm}^3$

Example

Find the volume of the conoid with a height of 4 cm and a base radius of 3 cm.

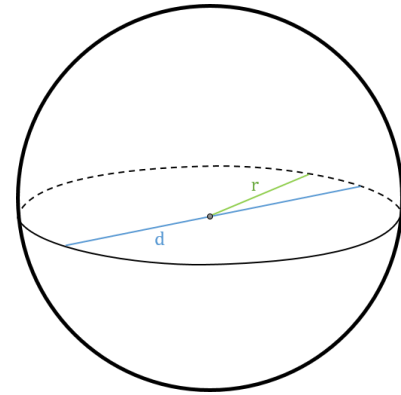
Conoid is a solid with a circular base and every section perpendicular to the fixed diameter is an isosceles triangle with constant height h .



- $r = 3 \text{ cm}$
- $h = 4 \text{ cm}$
- Middle part is an ellipse.
- By similar triangles, the smaller radius is half of r .
- $L = h$
- $V = \frac{h}{6}(A_1 + 4A_m + A_2)$
- $V = \frac{h}{6}\left(\pi r^2 + 4\left(\pi r \left(\frac{r}{2}\right)\right) + 0\right)$
- $V = \frac{h}{6}(\pi r^2 + 2\pi r^2)$
- $V = \frac{1}{2}\pi r^2 h$
- $V = \frac{1}{2}\pi(3)^2(4)$
- $V = 18\pi \text{ cm}^2 = 56.55 \text{ cm}^2$

V) Spheres and Parts of a Sphere

Sphere



- A solid generated by revolving a circle about its diameter.
- $V = \frac{4}{3}\pi r^3 = \frac{\pi}{6}d^3$
- $S_T = 4\pi r^2 = \pi d^2$

Example

What would happen to the volume of a sphere if the radius were tripled?

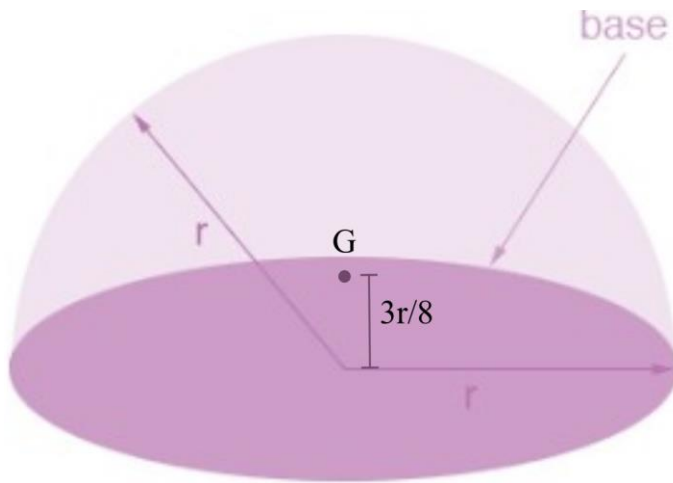
- $r = 3r_o$
- $V_o = \frac{4}{3}\pi r_o^3$
- $V = \frac{4}{3}\pi r^3$
- $V = \frac{4}{3}\pi(3r_o)^3$
- $V = 27\left(\frac{4}{3}\pi r_o^3\right)$
- $V = 27V_o$
- Volume **will increase 27-folds** (or will be multiplied 27).

Example

What is the ratio of the surface area of a sphere to its volume?

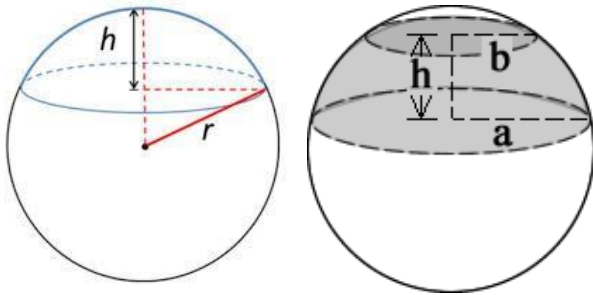
- $\frac{S_T}{V} = ?$
- $\frac{S_T}{V} = \frac{4\pi r^2}{\frac{4}{3}\pi r^3}$
- $\frac{S_T}{V} = \frac{3}{r}$

Hemisphere



- Half of a sphere
- $V = \frac{1}{2}V_{\text{sphere}} = \frac{2}{3}\pi r^3$
- $S_L = \frac{1}{2}S_{T,\text{sphere}} = 2\pi r^2$
- $S_T = S_L + A_B = 3\pi r^2$

Spherical Segment and Zone



- Zone, Z
 - Lateral area of the spherical segment
- One base (aka Spherical Cap, Left)
 - $Z = 2\pi r h$
 - $V = \frac{1}{3}\pi h^2(3r - h)$
- Two bases (aka Spherical Frustum, Right)
 - $V = \frac{1}{6}\pi h(3a^2 + 3b^2 + h^2)$

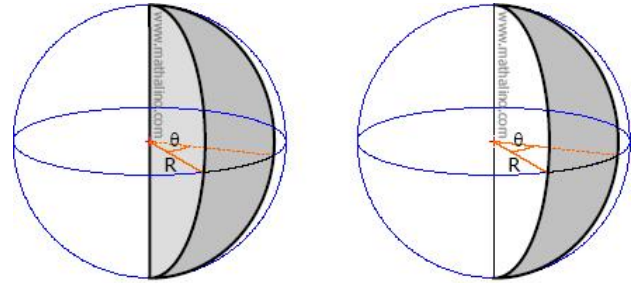
Example

The water in a hemispherical bowl 42 cm across the top, is 9 cm deep. How much more water is needed to fill the bowl to the brim?

- $r = \frac{42 \text{ cm}}{2} = 21 \text{ cm}$
- $h = 9 \text{ cm}$
- $V = V_{\text{hemisphere}} - V_{\text{water}}$

- $V = \frac{2}{3}\pi r^3 - \frac{1}{3}\pi r^2(3r - h)$
- $V = \frac{2}{3}\pi(21)^3 - \frac{1}{3}\pi(21)^2(3 \cdot 21 - 9)$
- $V = 4716\pi \text{ cm}^3 = 14815.75 \text{ cm}^3$

Lune and Spherical Wedge



Spherical Wedge

Spherical Lune

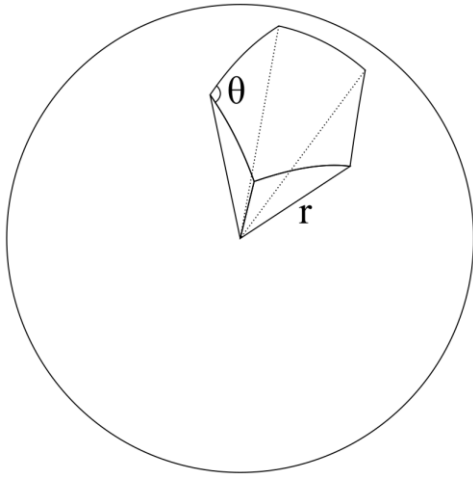
- Lune, A_L
 - Base Area of Spherical Wedge
- $L = \frac{\theta}{360^\circ}(4\pi r^2) = \frac{\theta}{90^\circ}\pi r^2$
- $V = \frac{\theta}{360^\circ}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}A_L r = \frac{\theta}{270^\circ}\pi r^3$

Example

Find the area of lune and volume of spherical wedge with a wedge angle of 80° taken from a sphere with radius equal to 60 cm.

- $r = 60 \text{ cm}$
- $\theta = 80^\circ$
- $A_L = ?$
 - $A_L = \frac{\theta}{360^\circ}(4\pi r^2)$
 - $A_L = \frac{80^\circ}{360^\circ}(4\pi(60)^2)$
 - $A_L = 10053.10 \text{ cm}^2 = 3200\pi \text{ cm}^2$
- $V = ?$
 - $V = \frac{\theta}{360^\circ}\left(\frac{4}{3}\pi r^3\right)$
 - $V = \frac{80^\circ}{360^\circ}\left(\frac{4}{3}\pi(60)^3\right)$
 - $V = 201061.93 \text{ cm}^3 = 64000\pi \text{ cm}^3$

Spherical Polygon and Spherical Pyramid



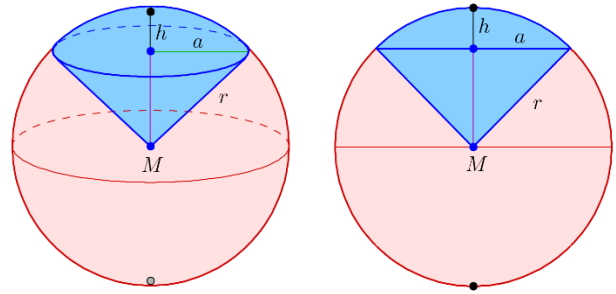
- **Spherical Polygon**
 - Polygon on a surface of a sphere made by great circles
 - Great circles – circles with the same center as the sphere
- $A_{SP} = E_{rad} r^2 = \frac{\pi}{180^\circ} E^\circ r^2$
 - $E_{rad} = \Sigma \theta_{rad} - (n - 2)\pi$
 - $E^\circ = \Sigma \theta^\circ - (n - 2)180^\circ$
- $V_{SP} = \frac{1}{3} A_{SP} r$
- $V_{SP} = \frac{1}{3} E_{rad} r^3 = \frac{1}{3} \frac{\pi}{180^\circ} E^\circ r^3$

Example

Find the area of spherical triangle with angles 50° , 80° , and 105° taken from a sphere with radius 60 cm. Find also the volume of the spherical pyramid formed.

- $\Sigma \theta^\circ = 50^\circ + 80^\circ + 105^\circ = 235^\circ$
- $n = 3$
- $r = 60 \text{ cm}$
- $A_{SP} = ?$
 - $A_{SP} = E_{rad} r^2 = \frac{\pi}{180^\circ} E^\circ r^2$
 - $E = \Sigma \theta^\circ - (n - 2)180^\circ$
 - $E = 235^\circ - 180^\circ = 55^\circ$
 - $A_{SP} = \frac{\pi}{180^\circ} 55^\circ (60)^2$
 - $A_{SP} = 3455.75 \text{ cm}^2 = 1100\pi \text{ cm}^2$
- $V_{SP} = ?$
 - $V_{SP} = \frac{1}{3} E_{rad} r^3 = \frac{1}{3} \frac{\pi}{180^\circ} E^\circ r^3$
 - $V_{SP} = \frac{1}{3} \frac{\pi}{180^\circ} 55^\circ (60)^3$
 - $V_{SP} = 69115.04 \text{ cm}^3 = 22000\pi \text{ cm}^3$

Spherical Sector or Spherical Cone



- $Z = 2\pi r h$
- $V = \frac{1}{3} Z r = \frac{2}{3} \pi r^2 h$

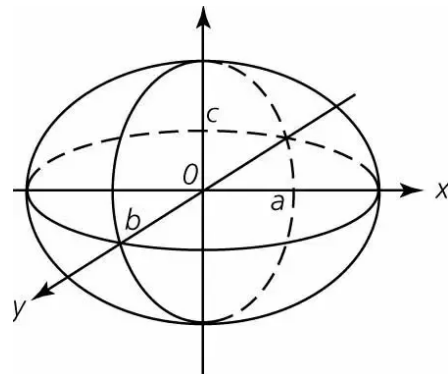
Example

Find the volume of spherical sector or spherical cone with a zone of height 20 cm and taken from a sphere with radius 60 cm.

- $h = 20 \text{ cm}$
- $r = 60 \text{ cm}$
- $V = ?$
 - $V = \frac{1}{3} Z r$
 - $V = \frac{1}{3} 2\pi r h \cdot r$
 - $V = \frac{2}{3} \pi (60)^2 (20)$
 - $V = 150796.45 \text{ cm}^3 = 48000\pi \text{ cm}^3$

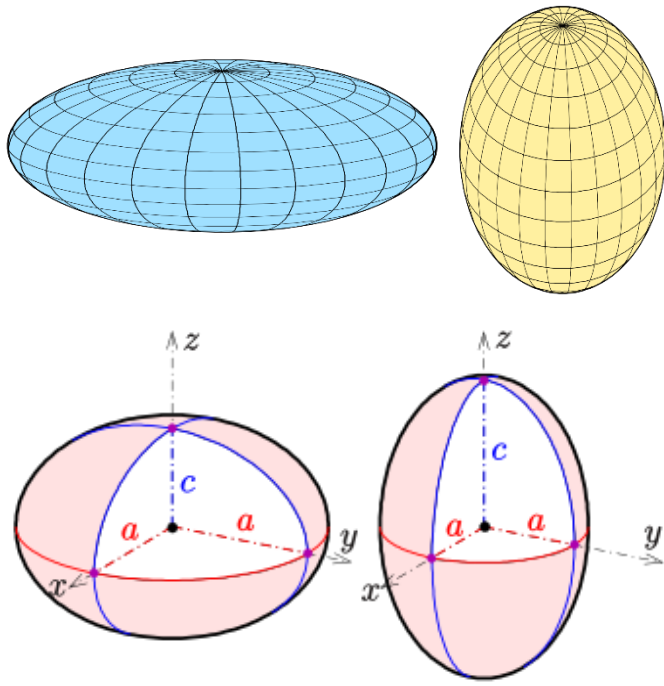
VI) The Ellipsoid / Spheroid

The Ellipsoid / Spheroid



- $V = \frac{4}{3} \pi a b c$

Special Cases



- Oblate (Left) – a solid generated when an ellipse is revolved at its minor axis
 - Resembles a mentos candy
 - One minor axis, two major axis radius
- Prolate (Right) – a solid generated when an ellipse is revolved about its major axis
 - Resembles a rugby ball
 - One major axis, two minor axis radius
- $V = \frac{4}{3}\pi a^2 c$

Inscribed Rectangular Parallelepiped in Ellipsoid

- The largest rectangular parallelepiped inscribed in the ellipsoid have...

Dimensions

$$L = \frac{2a}{\sqrt{3}}$$

$$W = \frac{2b}{\sqrt{3}}$$

$$H = \frac{2c}{\sqrt{3}}$$

Volume

$$V = LWH = \frac{8abc}{3\sqrt{3}}$$

Example

Find the volume of the solid generated by revolving an ellipse with semi-major axis of 4 cm and semi-minor axis of 3 cm about (a) the minor axis, (b) the major axis.

- (a)
 - One-minor axis, two major axis radius
 - $V = \frac{4}{3}\pi(4)^2(3)$
 - $V = 201.06 \text{ cm}^2 = 64 \pi \text{ cm}^2$
- (b)
 - One major axis, two minor axis radius
 - $V = \frac{4}{3}\pi(3)^2(4)$
 - $V = 150.80 \text{ cm}^2 = 48 \pi \text{ cm}^2$

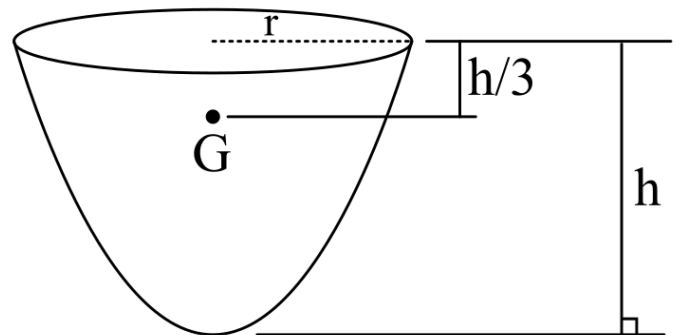
Example

Find the volume of the largest rectangular parallelepiped inscribed in an ellipsoid with $a = 4$, $b = 3$, and $c = 2$.

- $V = \frac{8}{3\sqrt{3}}abc$
- Substitute.
- $V = 36.95 \text{ cubic units}$

VII) The Paraboloid

The Paraboloid



- The solid generated when a parabolic segment is revolved about its axis

$$V = \frac{1}{2}\pi r^2 h$$

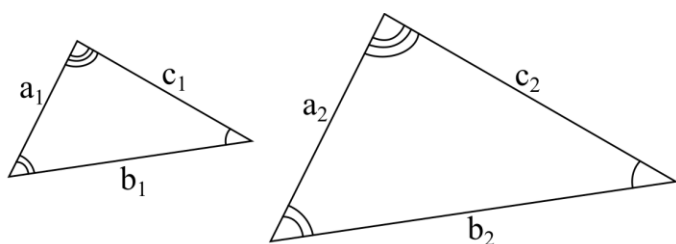
Example

Find the volume of the paraboloid with a radius of 3 cm and a height of 4 cm.

- $r = 3 \text{ cm}$
- $h = 4 \text{ cm}$
- $V = ?$
- $V = \frac{1}{2}\pi r^2 h$
- Substitute.
- $V = 56.55 \text{ cm}^2 = 18\pi \text{ cm}^2$

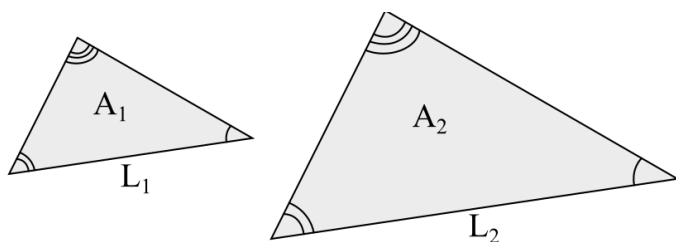
VIII) The Principle of Similar Figures

Line to Line



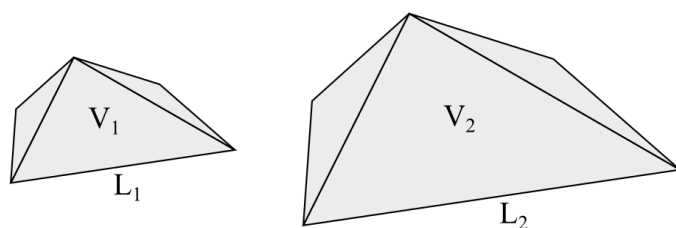
- $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

Length to Area



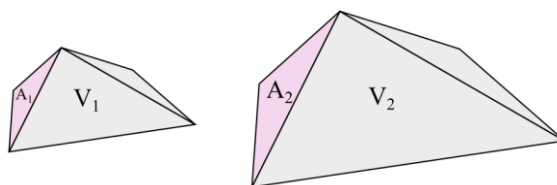
- $\frac{L_1}{L_2} = \sqrt{\frac{A_1}{A_2}}$

Length to Volume



- $\frac{L_1}{L_2} = \sqrt[3]{\frac{V_1}{V_2}}$

Area to Volume



- $\sqrt{\frac{A_1}{A_2}} = \sqrt[3]{\frac{V_1}{V_2}}$

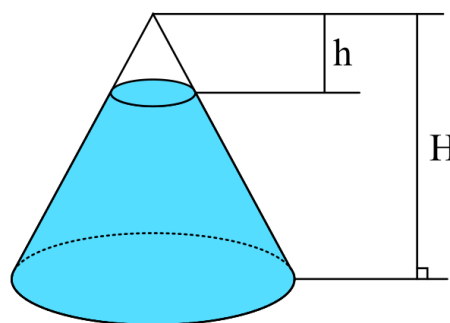
Example

Given are the sides of similar polygons which are 12 and 21 cm, respectively. Find the area of the largest polygon if the area of the small polygon is 64 sq. cm.

- $L_1 = 12 \text{ cm}$
- $L_2 = 21 \text{ cm}$
- $A_1 = 64 \text{ cm}^2$
- $A_2 = ?$
- $\frac{L_1}{L_2} = \sqrt{\frac{A_1}{A_2}}$
- Substitute and shift-solve.
- $A_2 = 196 \text{ cm}^2$

Example

A conical tank with its vertex on top is filled with water up to 2/3 of its height. Find the ratio of the volume of water to volume of tank.



- T denotes tank.
- W denotes water.
- E denotes empty space without water.
- $\frac{V_W}{V_T} = ?$
- $V_W = V_T$
- $V_W = V_T - V_E$
- $\sqrt[3]{\frac{V_E}{V_W}} = \frac{h}{H}$
- $\frac{V_E}{V_W} = \left(\frac{h}{H}\right)^3$

- $\frac{V_E}{V_W} = \left(\frac{\frac{1}{3}H}{H}\right)^3$
- $\frac{V_E}{V_W} = \frac{1}{27}$
- $\frac{V_W}{V_T} = \frac{V_T - V_E}{V_T}$
- $\frac{V_W}{V_T} = 1 - \frac{V_E}{V_T}$
- $\frac{V_W}{V_T} = 1 - \frac{1}{27}$
- $\frac{V_W}{V_T} = \frac{26}{27}$
- **$V_W:V_T; 26:27$**

Example

The capacities of two hemispherical tanks are in the ratio 64:125. If 4.8 kg of paint is required to pain the outer surface of the smaller tank, then how many kg of paint would be needed to paint the outer surface of the large tank.

- $\frac{V_1}{V_2} = \frac{64}{125}$
- $m_1 = 4.8 \text{ kg}$
- $m \propto S$ (directly proportional)
- $\sqrt{\frac{m_1}{m_2}} = \sqrt{\frac{S_1}{S_2}} = \sqrt[3]{\frac{V_1}{V_2}}$
- $\sqrt{\frac{4.8}{m_2}} = \sqrt[3]{\frac{64}{125}}$
- **$m_2 = 7.5 \text{ kg}$**

PLANE ANALYTIC GEOMETRY

plain analytic geometry

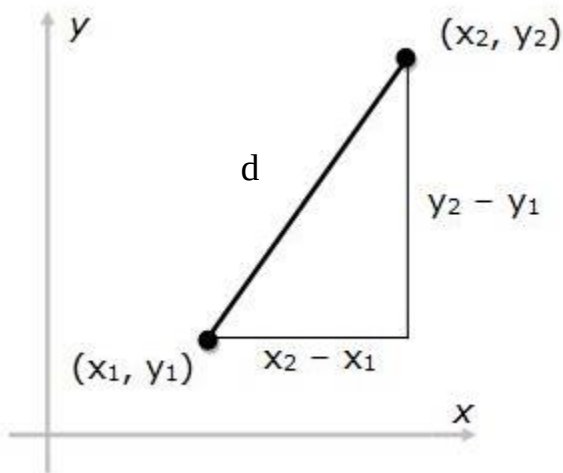
Plane Analytic Geometry

- Deals with:
 - Distance Between two points Knowing the coordinates, and
 - The equation and properties of a curve or locus of points

- (Alternative method)
- Use complex mode in calculator.
- Solve for $(3 + 2i) + (-2 - 6i)$
- Get the magnitude or radius r .
- $d = 9.43$ units**

I) Distance Between 2 Points

Distance Formula



- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- Abscissa
 - Another term for x-coordinate
 - Distance of a point from y-axis
- Ordinate
 - Another term for y-coordinate
 - Distance of a point from x-axis

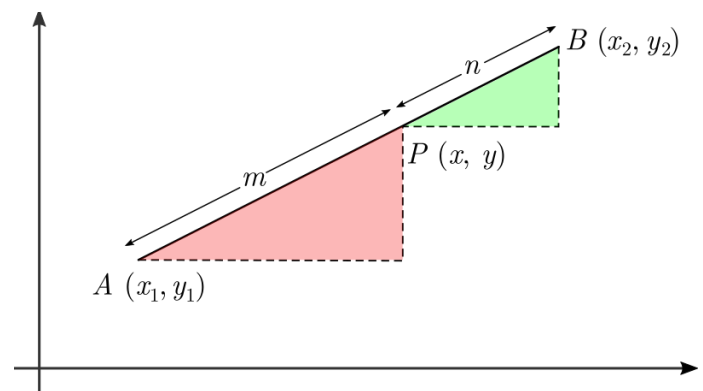
Example

Find the distance from point (3,2) to point (-2,-6).

- $(x_1, y_1) = (3, 2)$
- $(x_2, y_2) = (-2, -6)$
- $d = ?$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
- $d = \sqrt{(-2 - 3)^2 + (-6 - 2)^2}$
- $d = 9.43$ units**

II) The Division of Line Segment

Division of Line Segment



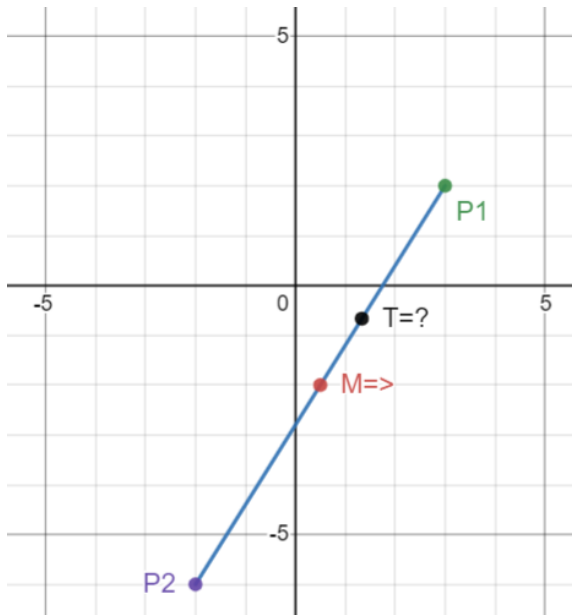
- Triangle with line AP is similar to Triangle with line AB
 - $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{m}{m + n} = \frac{AP}{AB} = r$
- $1 - r = \frac{n}{m + n}$
- $x = x_1 + r(x_2 - x_1) = \frac{mx_2 + nx_1}{m + n}$
- $y = y_1 + r(y_2 - y_1) = \frac{my_2 + ny_1}{m + n}$

Special Case: Middle Point M (x_m, y_m)

- $r = \frac{1}{2}$
- $x_m = \frac{1}{2}(x_1 + x_2)$
- $y_m = \frac{1}{2}(y_1 + y_2)$

Example

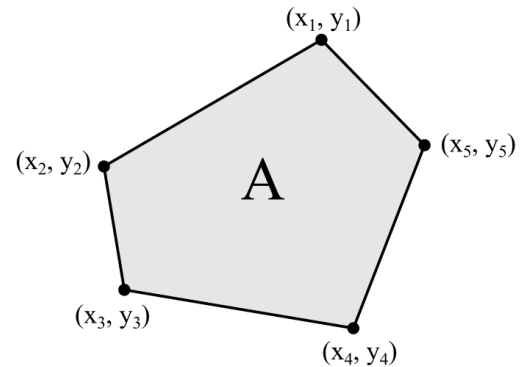
Given is a line segment from (3, 2) to (-2, -6). Find the coordinates of the first tri-section point from (3, 2) and the mid-point of the line segment.



- $(x_1, y_1) = (3, 2)$
- $(x_2, y_2) = (-2, -6)$
- $T(x, y) = \text{first tri-section point from } (3, 2) = ?$
 - $r = \frac{\overline{P_1 T}}{\overline{P_1 P_2}} = \frac{1}{3}$
 - $T(x, y) = T(x_1 + r(x_2 - x_1), y_1 + r(y_2 - y_1))$
 - $T(x, y) = T\left(3 + \frac{1}{3}(-2 - 3), 2 + \frac{1}{3}(-6 - 2)\right)$
 - $T(x, y) = \left(\frac{4}{3}, -\frac{2}{3}\right)$
- $M(x, y) = \text{mid-point} = ?$
 - $r = \frac{\overline{P_1 M}}{\overline{P_1 P_2}} = \frac{1}{2}$
 - $M(x, y) = M(x_1 + r(x_2 - x_1), y_1 + r(y_2 - y_1))$
 - $M(x, y) = M\left(3 + \frac{1}{2}(-2 - 3), 2 + \frac{1}{2}(-6 - 2)\right)$
 - $M(x, y) = \left(\frac{1}{2}, -2\right)$

III) Area of Polygon by Coordinates of Vertices Method

Vertices Method

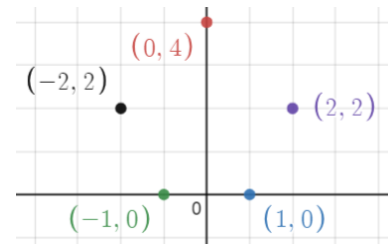


$$A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & \dots & x_n & x_1 \\ y_1 & y_2 & y_3 & \dots & y_n & y_1 \end{vmatrix}$$

- $A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_1 \end{vmatrix}$
- Vertices must be traced in **CCW direction** from vertex V1 (x1, y1).

Example

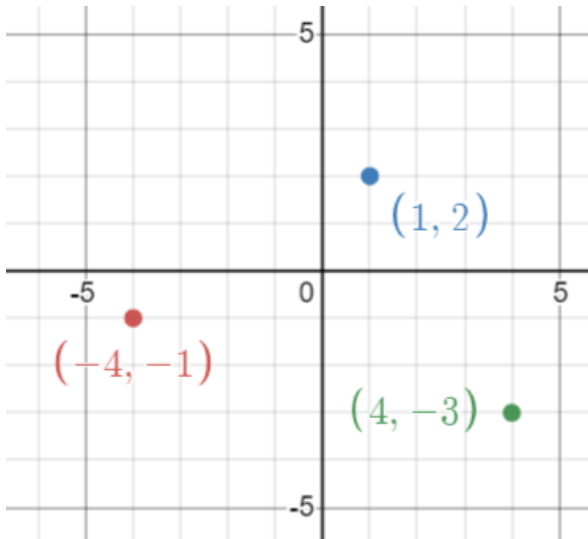
Find the area of pentagon with vertices at (1, 0), (-1, 0), (2, 2), (-2, 2), and (0, 4).



- Let us treat (0, 4) as vertex 1 and go counter-clockwise for vertex 2, 3, 4, and 5.
- $A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_1 \end{vmatrix}$
- $A = \frac{1}{2} \begin{vmatrix} 0 & -2 & -1 & 1 & 2 & 0 \\ 4 & 2 & 0 & 0 & 2 & 4 \end{vmatrix}$
- $A = \frac{1}{2} \{ +[0 \cdot 2 + (-2) \cdot 0 + (-1) \cdot 0 + 1 \cdot 2 + 2 \cdot 4] - [4 \cdot (-2) + 2 \cdot (-1) + 0 \cdot 1 + 0 \cdot 2 + 2 \cdot 0] \}$
- $A = \frac{1}{2} [(0 + 0 + 0 + 2 + 8) - (-8 - 2 + 0 + 0 + 0)]$
- $A = \frac{1}{2} (10 + 10)$
- **A = 10 cubic units**

Example

Find the area of the triangle with vertices at $-4 - i$, $1 + 2i$, and $4 - 3i$.

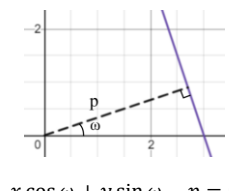


- Complex numbers in the complex plane can be treated as points in the cartesian plane.
- Let (1,2) be vertex 1. Vertex 2, and 3, is traced from vertex 1 counter-clockwise.
- $A = \frac{1}{2} \begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$
- $A = \frac{1}{2} \begin{vmatrix} 1 & -4 & 4 & 1 \\ 2 & -1 & -3 & 2 \end{vmatrix}$
- $A = \frac{1}{2} [+(-1 + 12 + 8) - (-8 - 4 - 3)]$
- $A = \frac{1}{2} [19 - (-15)]$
- A = 17 cubic units**
- Alternative method:
- Get the determinant of the ff using calculator.
- $A = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 2 & 1 \\ -4 & -1 & 1 \\ 4 & -3 & 1 \end{vmatrix}$
- A = 17 cubic units**

Properties and Equation of Line

Straight Line

Forms for the Equation of a Line

General Form	$Ax + By + C = 0$	$A > 0$
Slope-Intercept Form	$y = mx + b$	m is slope b is y-intercept
Point-Slope Form	$y - y_1 = m(x - x_1)$	(x_1, y_1) is a point in the line
Two-points Form	$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ or $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$	a is x-intercept
Intercept Form	$\frac{x}{a} + \frac{y}{b} = 1$	
Normal Form	 $x \cos \omega + y \sin \omega - p = 0$	p is perpendicular distance from origin to line. ω is angle to perpendicular distance

- From General Form to Normal Form
 - $\frac{Ax+By+C}{\pm\sqrt{A^2+B^2}} = 0$
 - Sign of radical is:
 - Opposite the sign of C
 - If C = 0, same as sign of B

Example

Find the equation of the line that bisects the 1st and 3rd quadrants.

- To bisect 1st and 3rd quadrants, a line must be drawn intersecting the origin.
 - $x_1 = 0$
 - $y_1 = 0$
- To bisect the 1st quadrant, from the origin, there must be equal rise and equal run.
 - $m = \frac{\text{rise}}{\text{run}} = 1$
- Alternatively
 - $m = \tan 45^\circ = 1$
- $y - y_1 = m(x - x_1)$
- $y = x$**

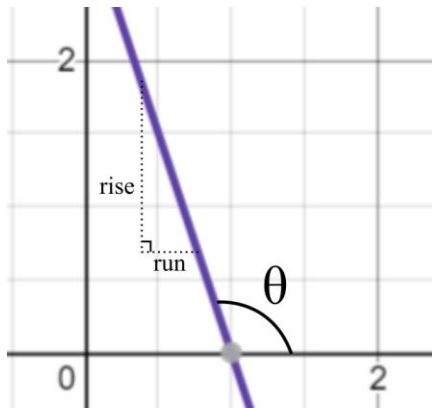
Example

Find the relation between °F and °C temperature scales and at what reading when they are equal.

For Water	x, T_C	y, T_F
Freezing Pt.	0	32
Boiling Pt.	100	212

- $(x_1, y_1) = (0, 32)$
- $(x_2, y_2) = (100, 212)$
- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{212 - 32}{100 - 0} = 1.8$
- $y - y_1 = m(x - x_1)$
- $y - 32 = 1.8(x - 0)$
- $y = 1.8x + 32$
- $T_F = 1.8T_C + 32$
- $T_F = T_C$
- $T_C = 1.8T_C + 32$
- $T_C = -40^\circ C$
- $T_F = -40^\circ F$

Properties of Lines (1): Slope, m



- $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$
- Run is negative in leftward direction.
- Rise is negative in downward direction.

Slope, m	Line Description
$m > 0$	Increasing, inclined to the right
$m < 0$	Decreasing, inclined to the left
$m = 0$	Horizontal
$m = \infty$	Vertical

Properties of Lines (2): Intersecting Lines

$$L_1: A_1x + B_1y + C_1 = 0$$

$$L_2: A_2x + B_2y + C_2 = 0$$

Description of Intersection	Formulas
$L_1 \parallel L_2$	$m_1 = m_2$ $\frac{A_1}{A_2} = \frac{B_1}{B_2}$
$L_1 \perp L_2$	$m_1 = -\frac{1}{m_2} \text{ or } m_2 = -\frac{1}{m_1}$ $A_1A_2 + B_1B_2 = 0$
L_1 coincides with L_2	$m_1 = m_2$ $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

- Parallel lines have same slopes.
- Perpendicular lines have slopes that are negative reciprocal of one another.

Properties of Lines (3): Distance from a Point

- Distance point $P_o(x_o, y_o)$ to line $Ax + By + C = 0$
- $d = \frac{Ax_o + By_o + C}{\pm \sqrt{A^2 + B^2}}$
 - Called **directed distance** – with sign
 - + when line L divides or separates origin and P_o
 - – when line L does not separate origin and P_o .
 - In other words, the origin and point P_o are on the same side of L.
- $|d| = \left| \frac{Ax_o + By_o + C}{\sqrt{A^2 + B^2}} \right|$
 - Magnitude Only

Properties of Lines (4): Distance Bet. Parallel Lines

- Distance between parallel lines
 - $L_1: Ax + By + C_1 = 0$
 - $L_2: Ax + By + C_2 = 0$
- $d = \left| \frac{C_2 - C_1}{\sqrt{A^2 + B^2}} \right|$

Example

Determine B such that $3x + 2y - 7 = 0$ is perpendicular to $2x - By + 2 = 0$.

- $A_1A_2 + B_1B_2 = 0$
- $3(2) + 2(-B) = 0$
- $B = 3$
- (Alternative Method)
- Transforming line 1 to point-slope form...
- $y = -\frac{3}{2}x + \frac{7}{2}$
- $m_1 = -\frac{3}{2}$
- Transforming line 2 to point-slope form...
- $y = \frac{2}{B}x + \frac{2}{B}$
- $m_2 = \frac{2}{B}$
- Perpendicular means slopes are negative reciprocal of each other.
- $m_1 = -\frac{1}{m_2}$
- $-\frac{3}{2} = -\frac{B}{2}$
- $B = 3$

Example

Points (6, -2) and (a, 6) are on the line with slope 4/3. Find the value of a.

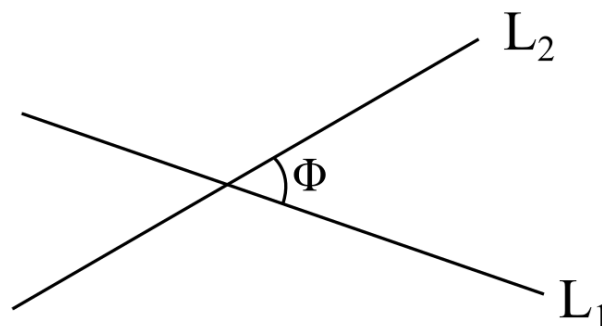
- Using formula for slope
- $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{6 - (-2)}{a - 6} = \frac{4}{3}$
- $a = 12$
- (Alternative Method)
- Using point-slope form...
- $y - (-2) = \frac{4}{3}(x - 6)$
- $y = \frac{4}{3}x - 8 - 2$
- $y = \frac{4}{3}x - 10$
- From (a, 6), a is the value of x when y = 6.
- $6 = \frac{4}{3}(a) - 10$
- $a = 12$

Example

Find the distance from the origin to the line $4x + 5y - 20 = 0$.

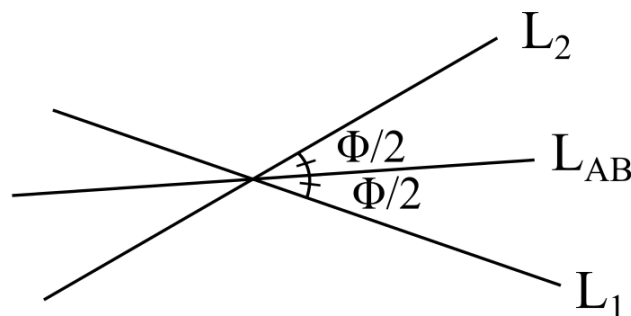
- $(x_o, y_o) = (0,0)$
- $A = 4; B = 5; C = -20$
- $d = \left| \frac{Ax_o + By_o + C}{\sqrt{A^2 + B^2}} \right|$
- Substitute.
- $d = 3.12$

Properties of Lines (5): Angle Bet. Two Lines



- $\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$
 - Angle is acute when right side is positive.
 - Angle is obtuse when right side is negative.
- $\phi = \arctan \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$
 - Formula for acute angle ϕ
- $\phi = \arccos \left| \frac{A_1 A_2 + B_1 B_2}{\sqrt{A_1^2 + B_1^2} \sqrt{A_2^2 + B_2^2}} \right|$

Properties of Lines (6): Eq'n of Angle Bisector



- $L_{AB}: \frac{A_1 x + B_1 y + C_1}{\pm \sqrt{A_1^2 + B_1^2}} = - \frac{A_2 x + B_2 y + C_2}{\pm \sqrt{A_2^2 + B_2^2}}$
 - Sign of radical is:
 - Opposite the sign of C
 - If C = 0, same as sign of B

Example

Find the distance between the lines $4x - 3y - 12 = 0$ and $4x - 3y + 8 = 0$.

- $A = 4$
- $B = -3$
- $C_1 = -12$
- $C_2 = 8$
- $d = \left| \frac{C_2 - C_1}{\sqrt{A^2 + B^2}} \right|$
- $d = \left| \frac{8 - (-12)}{\sqrt{4^2 + (-3)^2}} \right|$
- **$d = 4$ units**

The distance between the two lines is the perpendicular distance that is the same anywhere since the lines are parallel. Only distance between parallel lines can be determined. There is no distance between intersecting lines.

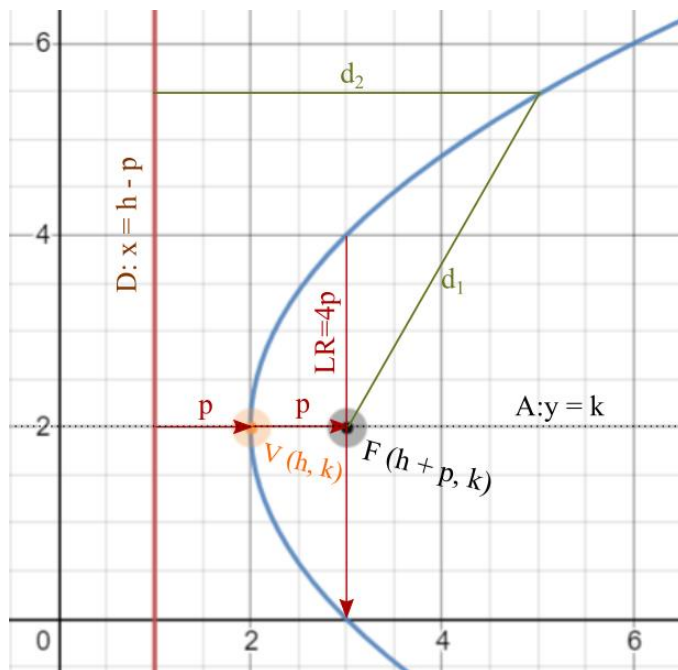
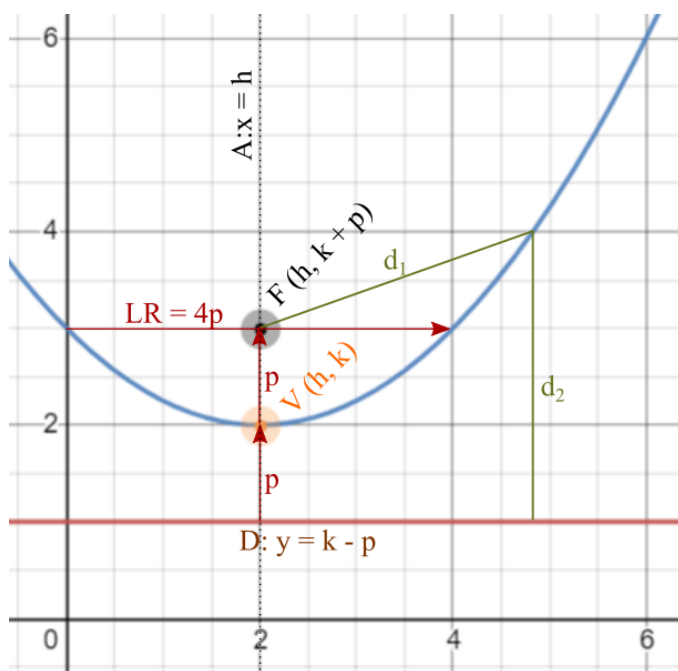
Example

The angle from L_1 , whose slope is $-1/3$ to the line L_2 is 135° . Find the slope of L_2 .

- $m_1 = -\frac{1}{3}$
- $m_2 = ?$
- $\phi = 135^\circ$
- $\tan \phi = \frac{m_2 - m_1}{1 + m_1 m_2}$
- $\tan 135^\circ = \frac{m_2 - (-\frac{1}{3})}{1 + (-\frac{1}{3})m_2}$
- Shift-solve.
- **$m_2 = -2$**

V) The Parabola

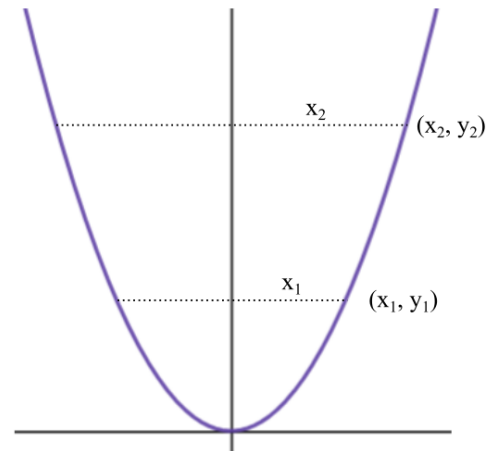
Parabola



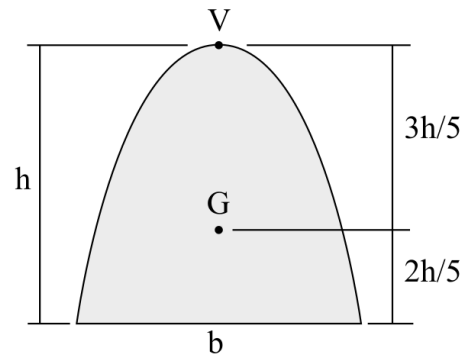
- 2nd degree equation
- Member of 2nd degree conic sections
- Locus of point with equal distances from a fixed point (focus, F) and a fixed line (directrix, D)
- $d_1 = d_2$
 - Focus to any point (distance) = Any point to directrix (distance)
- p is the distance from directrix to vertex or vertex to focus.
- The line with length $4p$ is called the latus rectum.

	Vertical	Horizontal
Gen. Form	$Ax^2 + Bx + Cy + D = 0$	$Ay^2 + By + Cx + D = 0$
Standard Form	$(x - h)^2 = 4p(y - k)$	$(y - k)^2 = 4p(x - h)$
Standard Form for V (0,0)	$x^2 = 4py$	$y^2 = 4px$
$p > 0$	Open upwards	Open rightwards
$p < 0$	Open downwards	Open leftwards
Length of Latus Rectum	$LR = 4p$	$LR = 4p$
	$LR = \left \frac{C}{A} \right $	$LR = \left \frac{C}{A} \right $
Vertex	$V(h, k)$	$V(h, k)$
Focus	$F(h, k + p)$	$F(h + p, k)$
Directrix	$D: y = k - p$	$D: x = h - p$
Axis	$x = h$	$y = k$
	Vertical Axis	Horizontal Axis
V (h,k) Formula	$h = -\frac{B}{2A}$	$k = -\frac{B}{2A}$
	$k = \frac{B^2 - 4AD}{4AC}$	$h = \frac{B^2 - 4AD}{4AC}$
	(negative bato)	(negative bato)
	$h = -\frac{B}{A2}$	$k = -\frac{B}{A2}$
	(bad four over AC)	(bad four over AC)
	$k = \frac{B^2 - AD4}{4AC}$	$h = \frac{B^2 - AD4}{4AC}$

- Squared Property of Parabola, SPP



- $\frac{y_1}{y_2} = \left(\frac{x_1}{x_2}\right)^2$
 - y and x is measured from vertex.
 - x can be the whole length of dashed line
- Parabolic Segment



- $A = \frac{2}{3}bh$

Example

Given is a parabola $x^2 - 4x - 4y = 0$. Find:

- The opening
- The location of vertex
- The location of focus
- The equation of directrix

- Transform general form of equation to standard form.
- $x^2 - 4x - 4y = 0$
- $x^2 - 4x = 4y$ (add $\left(\frac{4}{2}\right)^2$ both sides)
- $x^2 - 4x + 4 = 4y + 4$
- $(x - 2)^2 = 4(1)(y + 1)$
- $h = 2; k = -1; p = 1$
- $p > 0, x$ is squared

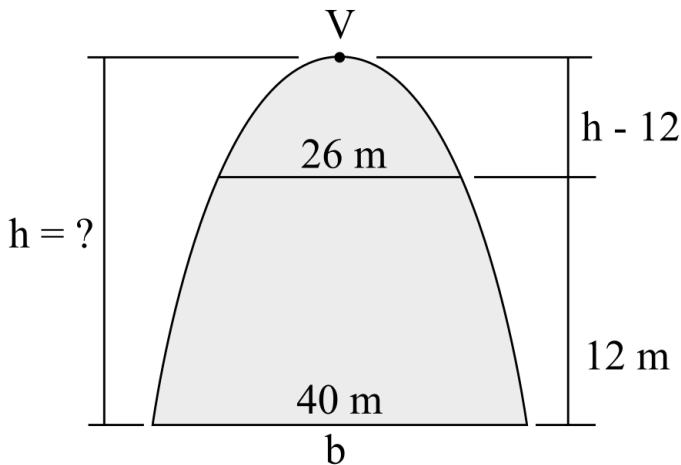
Properties of a Parabola

- Eccentricity, $e = \frac{d_1}{d_2} = 1$**
 - Distance of the point from the focus, d_1 divided by distance of the point from the directrix, d_2
 - d_1, d_2 is shown in figure on previous page.
 - Since $d_1 = d_2, e = 1$.
- Vertex to Focus Distance
 - p
- Length of Latus Rectum
 - $4p$

- (a) Open upwards
- (b) V (2, -1)
- $F(h, k + p)$
- (c) F (2, 0)
- $D: y = k - p$
- (d) $y = -2$
- (Alternative Method)
- Use formula for h and k using general form coefficients found in table.

Example

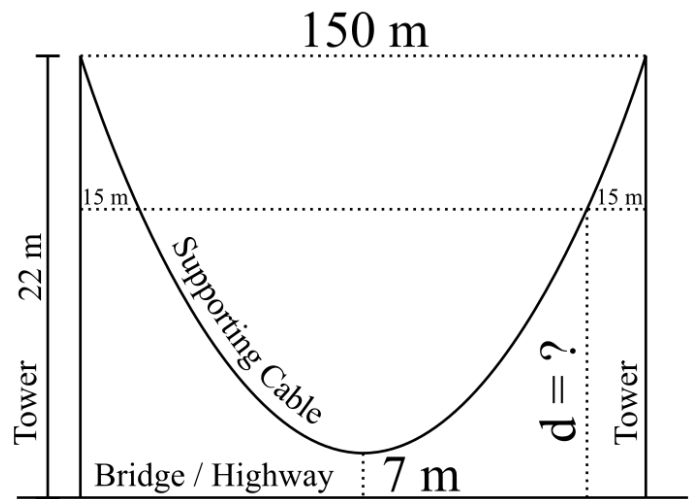
An arc in the form of parabolic curve is 40 m across the bottom. A flat horizontal beam 26 m long is placed 12 m above the base. Find the height of the arc.



- By SPP (Squared Property of Parabola) ...
- $\frac{h-12}{h} = \left(\frac{26 \div 2}{40 \div 2}\right)^2$
- Shift-solve.
- **$h = 20.78 \text{ m}$**

Example

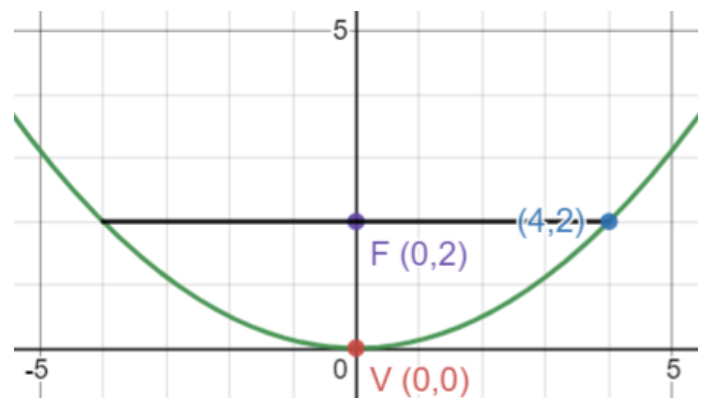
The supporting cable of a suspension bridge hangs in the form of a parabola from the top of 22 m tall towers which are 150 m horizontally apart. If the lowest point on the cable is 7 m above the roadway, find the vertical distance in meters of the cable from the roadway at the point which is 15 m from one of the supports.



- By SPP...
- $\frac{22-7}{d-7} = \left(\frac{150}{150-15-15}\right)^2$
- **$d = 16.6 \text{ m}$**

Example

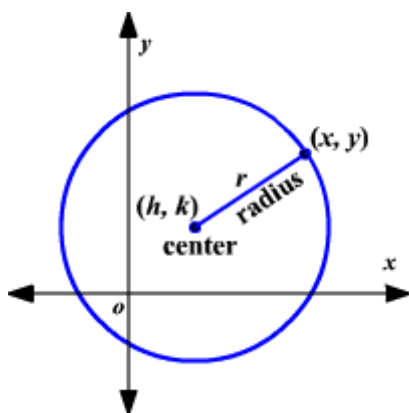
What is the area bounded by the curve $x^2 - 8y = 0$ and its latus rectum?



- $x^2 = 4(2)(y - 0)$
- $V(h, k) = (0, 0)$
- $p = 2$
- $b = LR = 4p = 8$
- $h = p = 2$
- $A = \frac{2}{3}bh$
- $A = \frac{2}{3}(8)(2)$
- **$A = \frac{32}{3} \text{ sq. units or } 10.67 \text{ sq. units}$**

VI) The Circle

Circle



- A 2nd degree conic section
- Locus of points with constant distance (radius) from a fixed point (center)

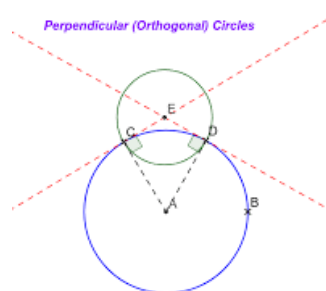
Circle

Gen. Form	$x^2 + y^2 + Ax + By + C = 0$
Standard (Center-Radius) Form	$(x - h)^2 + (y - k)^2 = r^2$
Standard Form with C (0,0)	$x^2 + y^2 = r^2$
Center	$C(h, k)$ $h = -\frac{A}{2}$ $k = -\frac{B}{2}$
Radius	$r = \frac{1}{2}\sqrt{A^2 + B^2 - 4C}$

Properties of a Circle

- $A = \pi r^2 = \frac{\pi}{4} d^2$
 - Area
- $C = 2\pi r = \pi d$
 - Circumference
- $d_T = \sqrt{x_o^2 + y_o^2 + Ax_o + By_o + C}$
 - Tangential distance from point P_o (x_o, y_o)
- (Circle 1 Eq.) – (Circle 2 Eq.) = 0
 - Equation of Radical Axis
 - Radical Axis

- Common chord or common tangent



- $A_1A_2 + B_1B_2 = 2(C_1 + C_2)$
 - Orthogonal Circles

Example

Given is a circle $x^2 + y^2 - 4x + 6y - 12 = 0$. Find (a) the location of center, (b) its radius, (c) the tangent distance from the point (8, 6)

- Transform to standard form.
- $(x^2 - 4x) + (y^2 + 6y) = 12$
- Divide middle coefficient by 2, then square.
- $(x^2 - 4x + 4) + (y^2 + 6y + 9) = 12 + 4 + 9$
- $(x - 2)^2 + (y + 3)^2 = 25 = 5^2$
- (a) C (2, -3)
- (b) r = 5 units
- $(x_o, y_o) = (8, 6)$
- $d_T = \sqrt{x_o^2 + y_o^2 - 4x_o + 6y_o - 12}$
- (c) $d_T = 2\sqrt{23}$ units or 9.59 units
- (Alternative Method)
- For a and b, transforming to standard form can not be done at all and just use the formulas
 - $h = -\frac{A}{2}$
 - $k = -\frac{B}{2}$
 - $r = \frac{1}{2}\sqrt{A^2 + B^2 - C^2}$

Example

Given are 2 circles $x^2 + y^2 + 2x - 3y - 5 = 0$ and $x^2 + y^2 + 4x + ky + 2 = 0$. Find the value of k so that they are orthogonal.

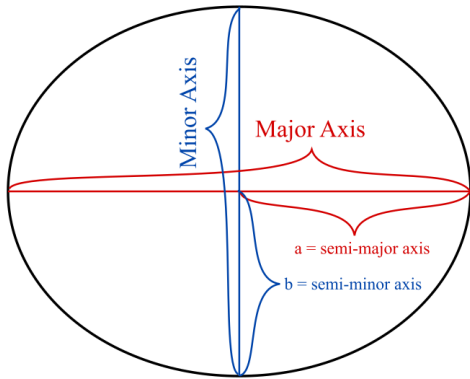
- $A_1A_2 + B_1B_2 = 2(C_1 + C_2)$
- $2 \cdot 4 + (-3) \cdot k = 2(-5 + 2)$
- $8 - 3k = -6$
- $k = \frac{14}{3}$

VII) The Ellipse

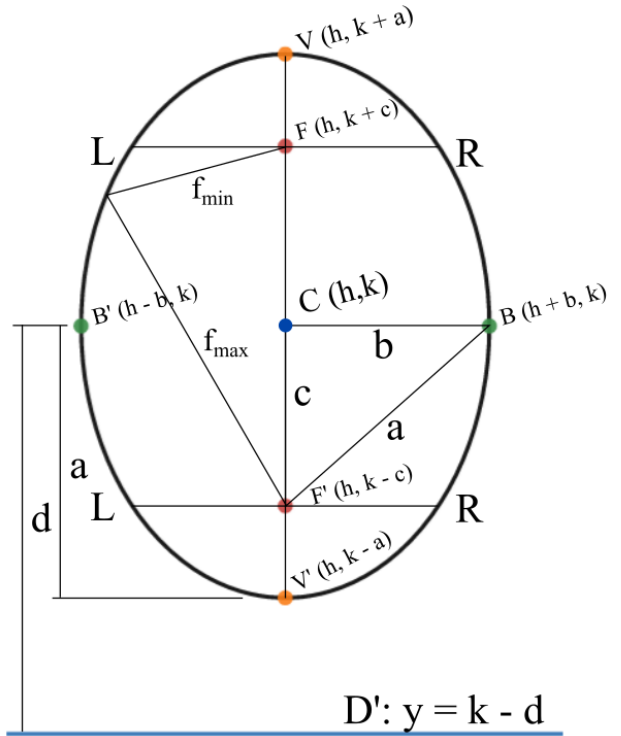
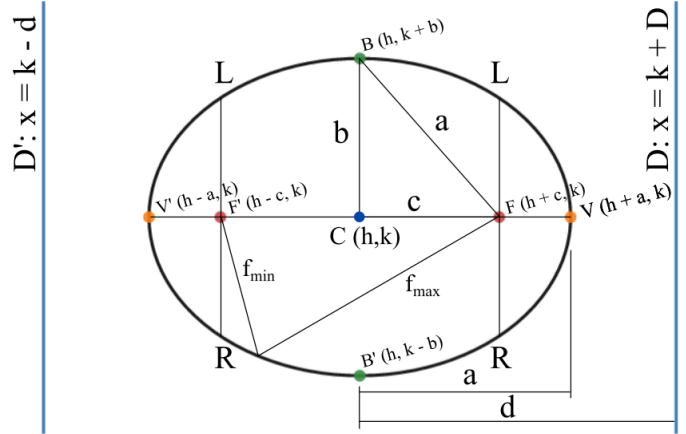
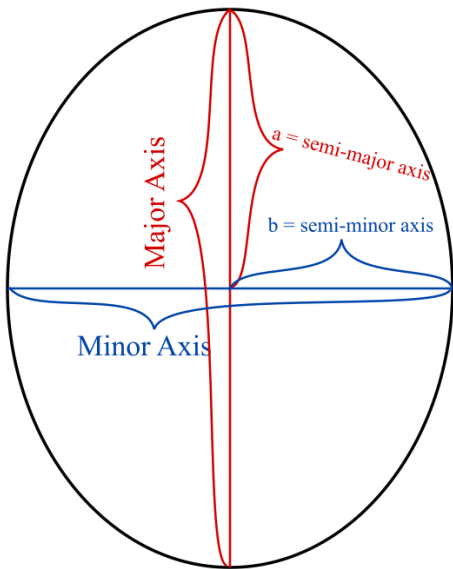
Ellipse

- A member of central conic section
- Locus of points which sum of distances from two fixed points (Foci F and F') is a constant.

Wide Ellipse



Tall Ellipse



	Wide Ellipse	Tall Ellipse
Gen. Form	$Ax^2 + By^2 + Cx + Dy + E = 0$	
A & B	$A < B$	$A > B$
Standard Form	$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$	$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$
Standard Form for V (0,0)	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$
a & b	$a > b$ $a = \sqrt{B}$ $b = \sqrt{A}$	$a > b$ $a = \sqrt{A}$ $b = \sqrt{B}$
Major Axis	Horizontal $MA = 2a$ (connects the vertices)	Vertical $MA = 2a$ (connects the vertices)
Minor Axis	Vertical $mA = 2b$ (connects the covertices)	Horizontal $MA = 2b$ (connects the covertices)
Center	$C(h, k)$ $h = -\frac{C}{2A}$ $k = -\frac{D}{2B}$	$C(h, k)$ $h = -\frac{C}{2A}$ $k = -\frac{D}{2B}$
Center to Foci Distance	$c = \sqrt{a^2 - b^2}$	$c = \sqrt{a^2 - b^2}$
Foci	$F(h + c, k)$ $F'(h - c, k)$	$F(h, k + c)$ $F'(h, k - c)$
Length of Latus Rectums	$LR = L'R'$ $= \frac{2b^2}{a}$	$LR = L'R'$ $= \frac{2b^2}{a}$
Vertices	$V(h + a, k)$ $V'(h - a, k)$	$V(h, k + a)$ $V'(h, k - a)$
Covertices	$B(h, k + b)$ $B'(h, k - b)$	$B(h + b, k)$ $B'(h - b, k)$
Distance of Directrices from center	$d = \frac{a^2}{c}$	$d = \frac{a^2}{c}$
Directrices	$D: x = k + d$ $D': x = k - d$	$D: y = k + d$ $D': y = k - d$

Properties of an Ellipse

- $e < 1$
 - First eccentricity
 - $e = \frac{c}{a} = \frac{a}{d}$
 - Second eccentricity
 - $e = \frac{c}{b}$
 - Third eccentricity
 - $e = \frac{c}{\sqrt{a^2 + b^2}}$
- $a^2 = b^2 + c^2$
 - a is semi-major axis
 - b is semi-minor axis
 - c is center to focus distance
- $LR = \frac{2b^2}{a}$
 - Length of latus rectums
- $d = \frac{a^2}{c} = \frac{a}{e}$
 - Distance of directrix from the center
- $A = \pi ab$
 - Area
- $P \approx 2\pi \sqrt{\frac{a^2 + b^2}{2}}$
- $f = a \pm ex$
 - Distance of point (x, y) on the ellipse from F and F'
 - $f_{max} = a + ex$
 - $f_{min} = a - ex$
 - $f_{max} + f_{min} = 2a$
 - Distance from the two foci to any point on the ellipse will always add up to 2a.

Example

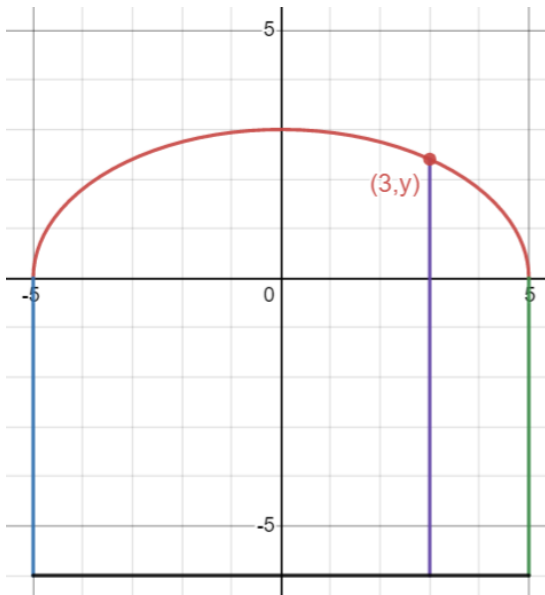
Given is an ellipse $9x^2 + 16y^2 - 36x + 32y - 92 = 0$. Find:

- The location of its center.
 - The length of its latus rectum.
 - The first eccentricity.
 - The area and perimeter.
- $9(x^2 - 4x + 4) + 16(y^2 + 2y + 1) = 92 + 36 + 16$
 - $9(x - 2)^2 + 16(y + 1)^2 = 144$

- $\frac{(x-2)^2}{16} + \frac{(y+1)^2}{9} = 1$
- (a) C (2, -1)
- $LR = \frac{2b^2}{a} = \frac{2(9)}{16}$
- (b) LR = 1.125
- $e = \frac{c}{a} = \frac{\sqrt{a^2-b^2}}{a}$
- $e = \frac{\sqrt{16-9}}{4} = 0.6614$
- (c) e = 0.6614
- $A = \pi ab$
- (d) A = 37.70 sq. units or 12π sq. units
- $P \approx 2\pi \sqrt{\frac{a^2+b^2}{2}}$
- (d) P = 22.21 units

Example

A hallway 10 m wide has a semi-elliptical arch ceiling which is 9 m high at the center from the floor. If the walls are 6 m tall at both ends of the ceiling, how high is the ceiling 2 m horizontally from either wall?

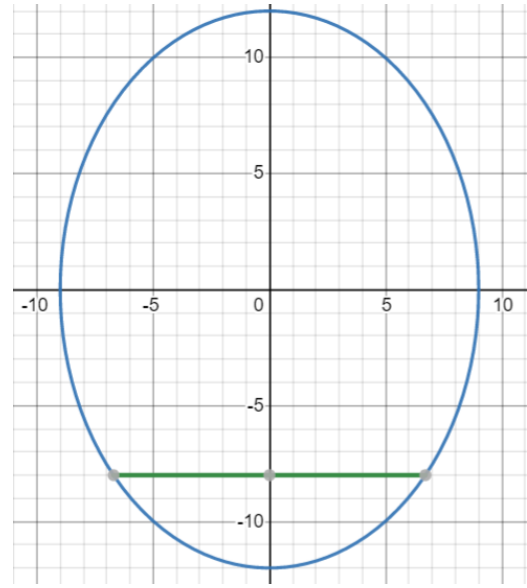


- Height of Ceiling from floor is 9m and height of walls are 6m. Thus, the height of semi-elliptical arch is 3m which is the b.
 - $b = 3$
- Since hallway is 10 m wide
 - $a = 5$
- $\frac{x^2}{25} + \frac{y^2}{9} = 1$
- 2 meters away from wall is 3 meters away from center.
- Getting y when x = 3...

- $y = 2.4$
- $H = 6 + y$ (required is the entire length of violet line)
- **H = 8.4**

Example

A trough has an elliptical cross-section which is 18 inches wide on top and 12 inches deep. If the water surface is 8 inches below the top, find the width of the water surface?



- $b = \frac{18}{2} = 9$
- $a = 12$
- $W = ?$ (green line length)
- $W = 2x$ when $y = -8$
- $\frac{x^2}{9^2} + \frac{y^2}{12^2} = 1$
- Get x when $y = -8$.
- $x = 6.71$
- $W = 2x$
- **W = 13.41 in or $6\sqrt{5}$ in**

Example

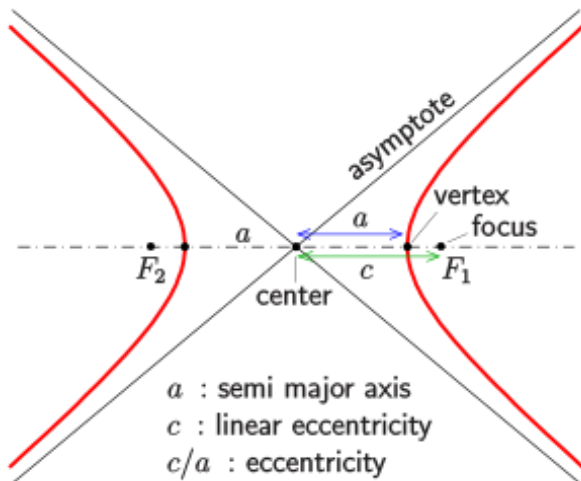
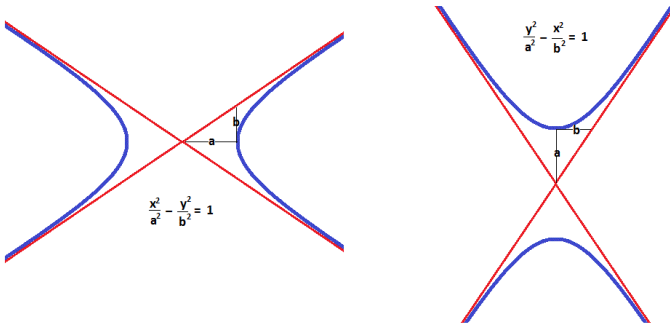
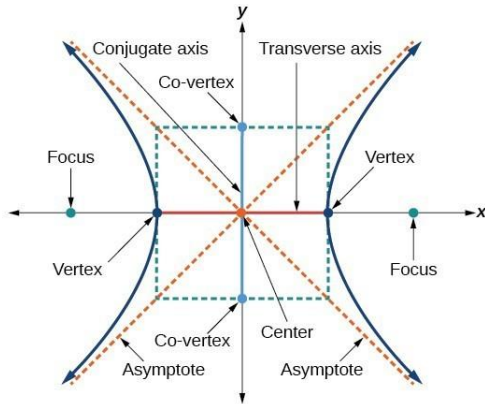
An ellipse has its center at (0, 0) with its axis horizontal. The distance between the vertices is 8 and its eccentricity is 0.5. Compute the length of the longest focal radius from point (2, 3) on the curve.

- $h = 0$
- $k = 0$
- $a = \frac{8}{2} = 4$
- $e = 0.5$
- $f_{max} = ?$

- $(x, y) = (2, 3)$
- $f_{max} = a + ex$
 - $4 + 0.5(2)$
 - 5
- $f_{max} = 5$

VIII) The Hyperbola

Hyperbola



- Is a member of the 2nd degree central conic sections with difference of distances from two fixed points (foci, F and F') are constant

	Horizontal	Vertical
Gen. Form	$Ax^2 + By^2 + Cx + Dy + E = 0$	
A & B	$B < 0$	$A < 0$
Standard Form	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$
Standard Form for V (0,0)	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$
	$a = \sqrt{ B }$ $b = \sqrt{A}$	$a = \sqrt{ A }$ $b = \sqrt{B}$
Transverse Axis	Horizontal $TA = 2a$ (connects the vertices)	Vertical $TA = 2a$ (connects the vertices)
Conjugate Axis	Vertical $CA = 2b$ (connects the covertices)	Horizontal $CA = 2b$ (connects the covertices)
Slope of Asymptotes	$m = \pm \frac{b}{a}$	$m = \pm \frac{a}{b}$
Center	$C(h, k)$ $h = -\frac{C}{2A}$ $k = -\frac{D}{2B}$	$C(h, k)$ $h = -\frac{C}{2A}$ $k = -\frac{D}{2B}$
Center to Foci Distance	$c = \sqrt{a^2 + b^2}$	$c = \sqrt{a^2 + b^2}$
Foci	$F(h + c, k)$ $F'(h - c, k)$	$F(h, k + c)$ $F'(h, k - c)$
Length of Latus Rectums	$LR = L'R' = \frac{2b^2}{a}$	$LR = L'R' = \frac{2b^2}{a}$
Vertices	$V(h + a, k)$ $V'(h - a, k)$	$V(h, k + a)$ $V'(h, k - a)$
Covertices	$B(h, k + b)$ $B'(h, k - b)$	$B(h + b, k)$ $B'(h - b, k)$
Distance of Directrices from center	$d = \frac{a^2}{c}$	$d = \frac{a^2}{c}$
Directrices	$D: x = k + d$ $D': x = k - d$	$D: y = k + d$ $D': y = k - d$

Properties of a Hyperbola

- $e > 1$
- $e = \frac{c}{a}$
- $c^2 = a^2 + b^2$
- $LR = \frac{2b^2}{a}$
- $d = \frac{a}{e}$
- $\phi = 2 \arctan\left(\frac{b}{a}\right)$
 - Angle between asymptotes wherein the hyperbola is found
- Eq. of asymptotes
 - Use point slope form of line.
 - The point is the center of the hyperbola.
- Point of intersection of asymptote is center of parabola.
- Slope of asymptote
 - $m = \pm \frac{b}{a}$ or $\pm \frac{a}{b}$

Example

Given is a hyperbola $x^2 - 3y^2 - 4x - 6y - 2 = 0$. Find:

- LR
- Eccentricity
- Angle between asymptotes
- Distance between directrices

- $A = 1, B = -3, C = -4, D = -6, E = -2$
- $B < 1 \rightarrow$ Horizontal Hyperbola
- $a = \sqrt{|B|} = \sqrt{3}$
- $b = \sqrt{A} = 1$
- $LR = \frac{2b^2}{a} = \frac{2(1)^2}{\sqrt{3}}$
- (a) $LR = 1.15$
- $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a} = \frac{\sqrt{3+1}}{\sqrt{3}}$
- (b) $e = 1.15$
- $\phi = \arctan\left(\frac{b}{a}\right) = 2 \arctan\left(\frac{1}{\sqrt{3}}\right)$
- (c) $\phi = 60^\circ$
- $D = 2d = 2\left(\frac{a^2}{c}\right) = 2\left(\frac{3}{\sqrt{3+1}}\right)$
- (d) $D = 3$

Example

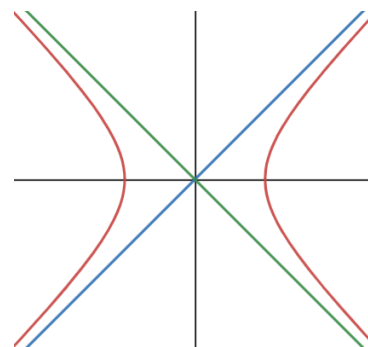
Find the equation of the hyperbola with center at (0, 0) and transverse axis 6 and a focus at (5, 0)

- $TA = 2a = 6 \rightarrow a = 3$
- $h = 0; k = 0$
- $c = 5$ (distance of center to focus)
- $c^2 = a^2 + b^2 \rightarrow b = 4$
- Transverse axis is horizontal.
- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- $\frac{x^2}{9} - \frac{y^2}{16} = 1$

Special Cases of Hyperbola

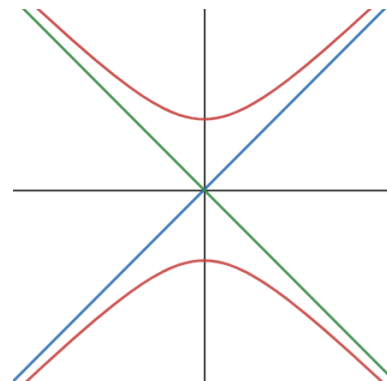
Equilateral “Rectangular” Hyperbola

- $a = b$
- Horizontal Transverse Axis



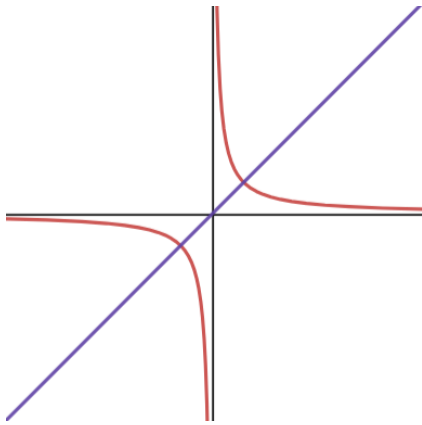
◦ $x^2 - y^2 = a^2$

- Vertical Transverse Axis

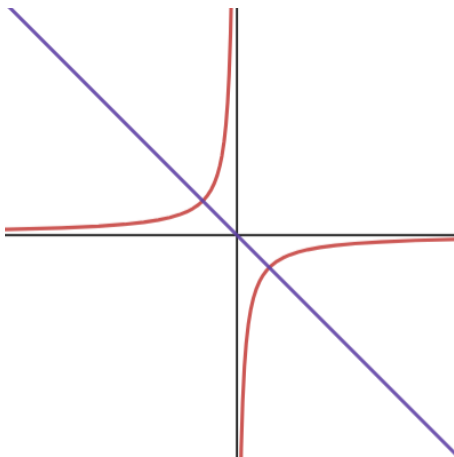


◦ $y^2 - x^2 = a^2$

- Diagonal Transverse Axis (Leaning Right)



- $xy = \frac{1}{2}a^2$
- Asymptotes is x and y axis.
- Diagonal Transverse Axis (Leaning Left)



- $xy = -\frac{1}{2}a^2$
- Asymptotes is x and y axis.

Example

Find the vertices and eccentricity of the equilateral hyperbola $xy = 4$

- $\frac{1}{2}a^2 = 4$
- $a^2 = b^2 = 2$
- Diagonal transverse axis leaning to the right.
- Transverse axis is $y = x$
- Solving for intersection.
- $x^2 = 4$
- $x = \pm 2$
- $y = \pm 2$
- (a)
 $V(2, 2)$
 $V'(-2, -2)$
- $e = \frac{c}{a}$

- $e = \frac{c}{a} = \frac{\sqrt{a^2+b^2}}{a} = \frac{\sqrt{2+2}}{\sqrt{2}}$
- (b) $e = 1.41 = \sqrt{2}$

IX) General Equation of 2nd Degree Conic Sections

General Equation of 2nd Degree Conic Sections

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

	Condition	Eccentricity
Circle	$A = C, B = 0$	$e = 0$
Parabola	$B^2 - 4AC = 0$	$e = 1$
Ellipse	$B^2 - 4AC < 0$	$e < 1$
Hyperbola	$B^2 - 4AC > 0$	$e > 1$

- $D = B^2 - 4AC$
 - Discriminant or Invariant
- Angle of rotation to eliminate the product term xy is:
 - $\tan 2\theta = \frac{B}{A-C}$
 - $0^\circ < 2\theta < 180^\circ$
 - $0^\circ < \theta < 90^\circ$

Example

Given equation is $2x^2 - \sqrt{3}xy + y^2 = 2$

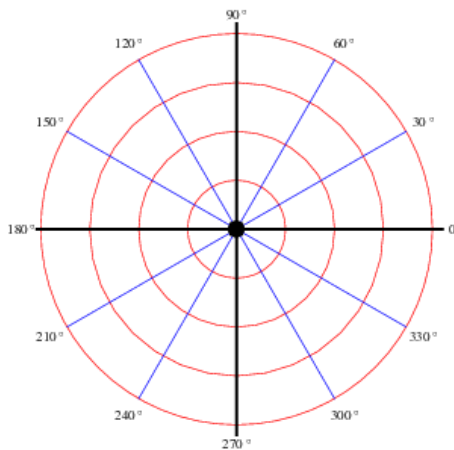
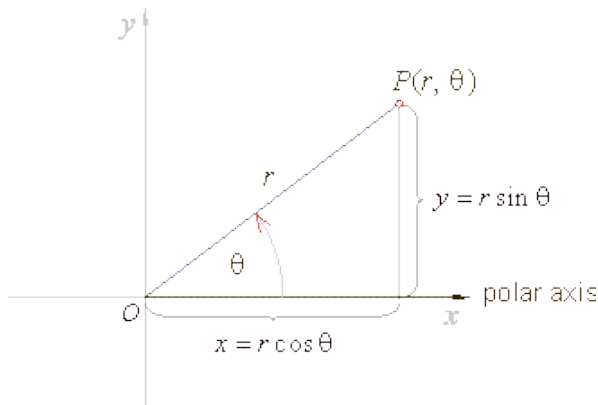
- (a) Identify the curve.
- (b) To what angle should the axis be rotated to eliminate the product term xy .

- $A = 2, B = -\sqrt{3}, C = 1$
- $D = B^2 - 4AC$
- $D = 3 - 4(2)(1)$
- $D = -5 < 0$
- **Ellipse.**
- $\tan 2\theta = \frac{B}{A-C}$
- $2\theta = \arctan\left(\frac{B}{A-C}\right)$
- $2\theta = \arctan\left(\frac{-\sqrt{3}}{2-1}\right)$
- $2\theta = -60^\circ$
 - Fourth quadrant.
- But $0^\circ < 2\theta < 180^\circ \dots$
- Tangent is also negative in second quadrant.

- The angle in second quadrant can be taken by extending terminal of angle -60° in the fourth quadrant to second quadrant.
- $2\theta = 180^\circ - 60^\circ = 120^\circ$
- **(b) $\theta = 60^\circ$**

X) The Polar Coordinate System

Polar Coordinate System



- $x = r \cos \theta$
- $y = r \sin \theta$
- $r = \sqrt{x^2 + y^2}$
- $\theta = \arctan \frac{y}{x} + \phi$
 - $\phi = 0$ if $x > 0$
 - $\phi = 180^\circ = \pi$ if $x < 0$
- r is radius vector
- θ is vectorial angle
 - $-180^\circ < \theta < 180^\circ$
- X-axis is polar axis.
- Y-axis is normal axis.

Example

Find the equivalent of $(-3, 4)$ in the polar coordinate system.

- $x = -3, y = 4$
- $r = \sqrt{x^2 + y^2} = \sqrt{3^2 + 4^2}$
- $r = 5$
- $\theta = \arctan \frac{y}{x} + \phi = \arctan \frac{4}{-3} + 180^\circ$
- $\theta = 126.87^\circ$
- **$(-3, 4) = (5, 126.87^\circ)$**

Example

Find the equivalent equation of the circle $x^2 + y^2 - 2y = 0$ in the polar coordinate system.

- $r^2 = x^2 + y^2$
- $y = r \sin \theta$
- $r^2 - 2r \sin \theta = 0$
- **$r = 2 \sin \theta$**

Example

Identify the curve represented by:

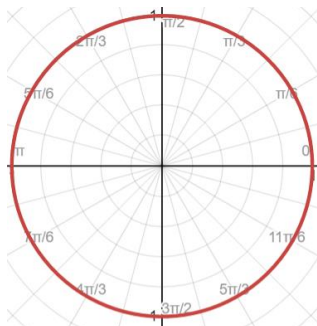
$$r = \frac{2}{1 + \cos \theta}$$

- $\cos \theta = \frac{x}{r}$
- $r = \frac{2}{1 + \frac{x}{r}}$
- $r = \frac{2r}{r+x}$
- $1 = \frac{2}{r+x}$
- $r + x = 2$
- $\sqrt{x^2 + y^2} + x = 2$
- $\sqrt{x^2 + y^2} = 2 - x$
- $x^2 + y^2 = 4 - 4x + x^2$
- $y^2 = 4(x - 1)$
- **Parabola.**

XI) Common Polar Curves

Circles

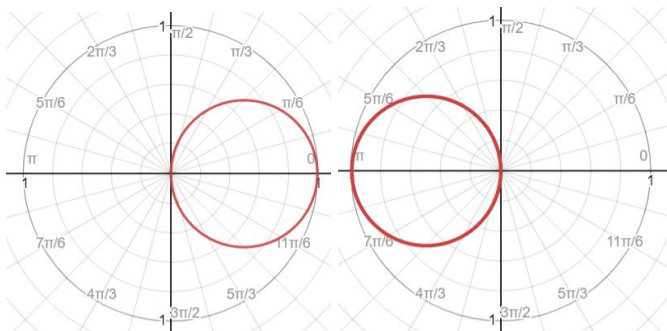
- Circle with radius = a and center at $(0, 0)$



- $r = a$

- Radius is a .

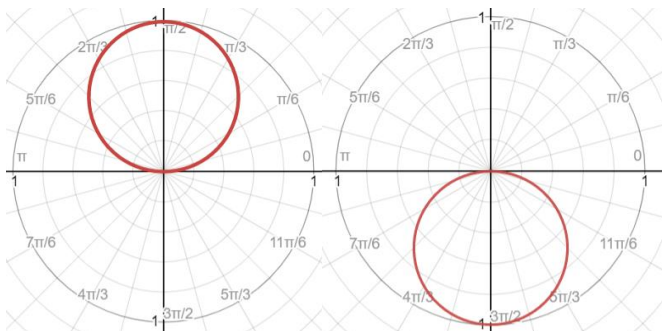
- Circle with center on x-axis and tangent to y-axis.



- $r = a \cos \theta$

- $a > 0$ circle at right side of y-axis
- $a < 0$ circle at left side of y-axis
- Radius is $a/2$.

- Circle with center on y-axis and tangent to x-axis.

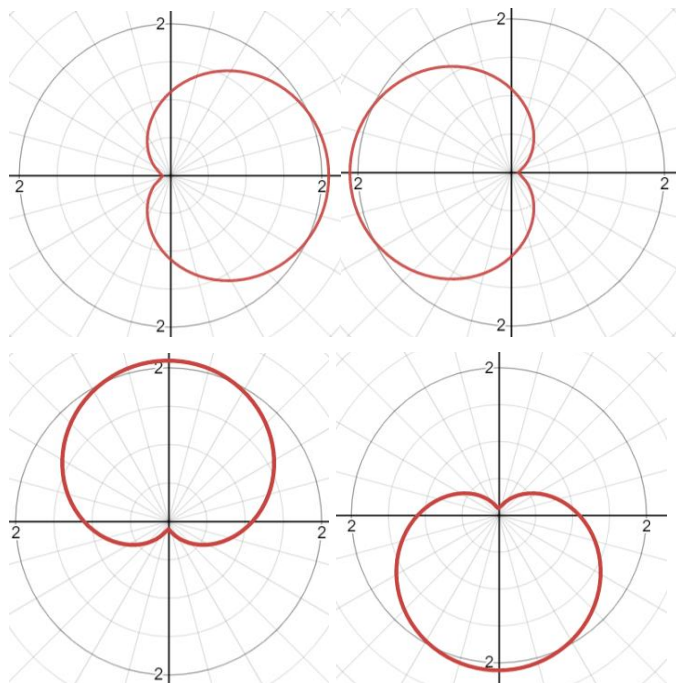


- $r = a \sin \theta$

- $a > 0$ circle at top side of x-axis
- $a < 0$ circle at bottom side of x-axis
- Radius is $a/2$.

Limacon of Pascal

- $a > b$



- $r = a + b \cos \theta$

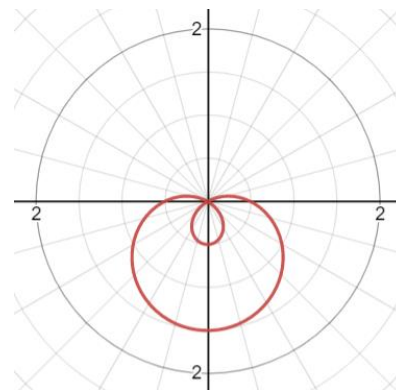
- $b > 0$ heart bottom points right
- $b < 0$ heart bottom points left

- $r = a + b \sin \theta$

- $b > 0$ heart bottom points up
- $b < 0$ heart bottom points down

- $A = \frac{\pi}{2} (2a^2 + b^2)$

- $a < b$

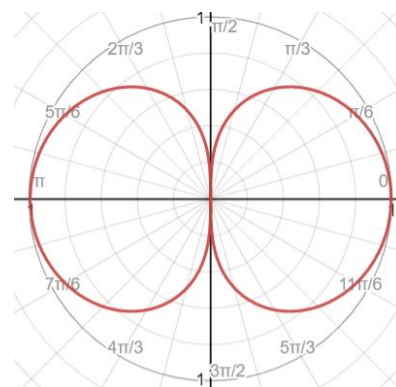


- Will have inner loops.
- Intersect itself at origin.

Cardioid

- Similar to limaçon if $a = b$
- $r = a(1 \pm \cos \theta)$
- $r = a(1 \pm \sin \theta)$
- Has its cusp on the origin.
 - The point where the curve changes point abruptly.
- Can be pictured the same way as limaçons with the cusp at the origin.
- $A = \frac{3\pi}{2}a^2$
- $P = 8a$

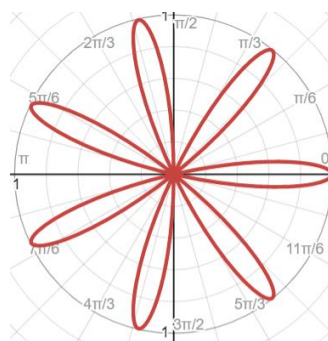
- $r^2 = a^2 \cos \theta$



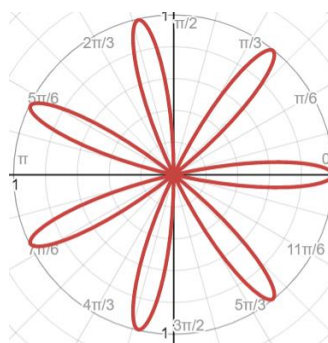
- $A = 2a^2$

Rose

- $r = a \cos(n\theta), n = 2, 3, 4$



- $r = a \sin(n\theta), n = 2, 3, 4$



- Number of petals and Area

- n is odd

- Number of petals is n

- $A = \frac{\pi}{4}a^2$

- n is even

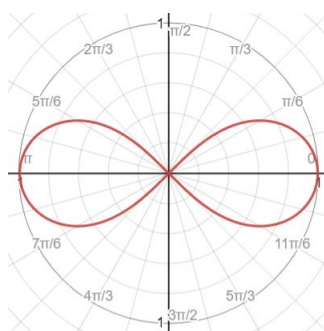
- Number of petals is $2n$

- $A = \frac{\pi}{2}a^2$

- $n = 1$ it becomes a circle.

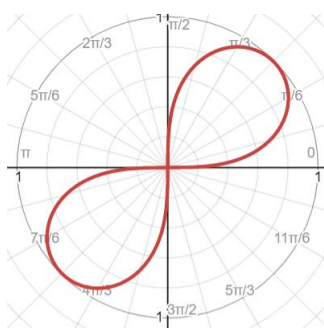
Lemniscate of Bernoulli

- $r^2 = a^2 \cos(2\theta)$



- $A = a^2$

- $r^2 = a^2 \sin(2\theta)$



- $A = a^2$

Example

Find the area bounded by the following polar curves.

- (1) $r = 2(1 + \cos \theta)$
- (2) $r = 2 \sin(2\theta)$
- (3) $r = 2 \cos \theta$
- (4) $r = 2 + \cos \theta$
- (5) $r^2 = 16 \sin(2\theta)$
- (6) $r = 1 - \sin \theta$

- First, identify the type of curve, and then use formula of area for that type of curve.

- (1) is a cardioid.

$$A = \frac{3\pi}{2} a^2 = \frac{3\pi}{2} (2)^2$$

$$A = 6\pi \text{ sq. units}$$

- (2) is a rose with even n.

$$A = \frac{\pi}{2} a^2$$

$$A = \frac{\pi}{2} (2)^2$$

$$A = 2\pi \text{ sq. units}$$

- (3) is a circle with radius a/2

$$A = \frac{\pi a^2}{4} = \frac{\pi(2)^2}{4}$$

$$A = \pi \text{ sq. units}$$

- (4) is a Limacon

$$A = \frac{\pi}{2} (2a^2 + b^2)$$

$$A = \frac{\pi}{2} (2(2)^2 + (1)^2)$$

$$A = 4.5\pi \text{ sq. units}$$

- (5) is a lemniscate.

$$A = a^2 = (16)$$

$$A = 16 \text{ sq. units}$$

- (6) is a cardioid.

$$A = \frac{3\pi}{2} a^2$$

$$A = \frac{3\pi}{2} (1)^2$$

$$A = 1.5\pi \text{ sq. units}$$

XII) Equation of Curves in Complex Variable Z

Convert Complex Number from Rectangular Form to Polar (Trigonometric) Form

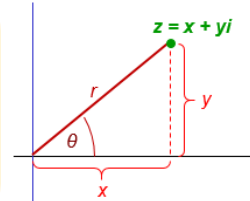
$$z = x + yi \text{ (rectangular form)}$$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = r(\cos \theta + i \sin \theta) \text{ (polar form)}$$



Example

Identify the curve and find the area bounded by the curve $|z - 2| = 3$

$$|z - 2| = |x + jy - 2|$$

$$\sqrt{(x - 2)^2 + y^2} = 3$$

$$(x - 2)^2 + y^2 = 9$$

- Circle.

$$r^2 = 9$$

$$A = \pi r^2$$

$$A = 9\pi \text{ sq. units}$$

Example

Identify the curve represented by:

$$|z - 3| + |z + 3| = 10$$

$$z = x + jy$$

$$|x + jy - 3| + |x + jy + 3| = 10$$

$$\sqrt{(x - 3)^2 + y^2} + \sqrt{(x + 3)^2 + y^2} = 10$$

$$\sqrt{(x - 3)^2 + y^2} = 10 - \sqrt{(x + 3)^2 + y^2}$$

$$(x - 3)^2 + y^2 = 100 - 20\sqrt{(x + 3)^2 + y^2} + (x + 3)^2 + y^2$$

$$x^2 - 6x + 9 + y^2 = 100 - 20\sqrt{(x + 3)^2 + y^2} + x^2 + 6x + 9 + y^2$$

$$12x + 100 = 20\sqrt{(x + 3)^2 + y^2}$$

$$3x + 25 = 5\sqrt{(x + 3)^2 + y^2}$$

$$9x^2 + 150x + 25^2 = 25((x + 3)^2 + y^2)$$

$$= 25(x^2 + 6x + 9 + y^2)$$

$$25x^2 + 150x + 225 + 25y^2 - 9x^2 - 150x - 25^2 = 0$$

$$16x^2 - 400 + 25y^2 = 0$$

- $\frac{x^2}{25} - 1 + \frac{y^2}{16} = 0$
- $\frac{x^2}{25} + \frac{y^2}{16} = 1$
- **Ellipse.**

Note: If $|z - 3| - |z + 3| = 10$, curve is a hyperbola.

Example

Identify the curve represented by

$$\operatorname{Im}\{z^2\} = 4$$

- $\operatorname{Im}\{(x + jy)^2\} = 4$
- $\operatorname{Im}\{x^2 - y^2 + 2jxy\} = 4$
- $2xy = 4$
- $xy = 2$
- **Equilateral Hyperbola.**

SOLID ANALYTIC GEOMETRY

solid analytic geometry

I) Distance Between Two Points

Distance Between Two Points

- $P_1 (x_1, y_1, z_1)$
- $P_2 (x_2, y_2, z_2)$
- $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Example

Find the distance from (3, -2, 1) to (2, 4, -2).

- $d = \sqrt{(3 - 2)^2 + (-2 - 4)^2 + (1 - (-2))^2}$
- $d = 6.78 \text{ units}$

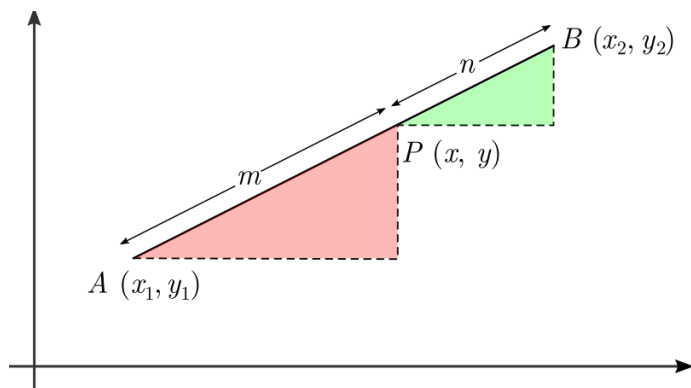
Example

Find the point on the line $x = y = z$ that is equidistant from the point (3,0,5) and (1,-1,4).

- $P_1 (3, 0, 5)$
- $P_2 (1, -1, 4)$
- $P (x, y, z) = P(x, x, x)$
- $P_1 P = P_2 P$
- $\sqrt{(x - 3)^2 + x^2 + (x - 5)^2} = \sqrt{(x - 1)^2 + (x + 1)^2 + (x - 4)^2}$
- Shift-solve.
- $x = 2$
- $P (2, 2, 2)$

II) Division of Line Segment

Division of Line Segment



- $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = \frac{m}{m+n} = \frac{\overline{AP}}{\overline{AB}} = r$
- $1 - r = \frac{n}{m+n}$
- $x = x_1 + r(x_2 - x_1)$
- $y = y_1 + r(y_2 - y_1)$
- $z = z_1 + r(z_2 - z_1)$

Special Case: Middle Point M (x_m, y_m)

- $r = \frac{1}{2}$
- $x_m = \frac{1}{2}(x_1 + x_2)$
- $y_m = \frac{1}{2}(y_1 + y_2)$
- $z_m = \frac{1}{2}(z_1 + z_2)$

Example

Find the mid-point of the line segment joining (3, 4, 2) and (1, 6, 3).

- $r = \frac{1}{2}$
- $x = \frac{1}{2}(3 + 1) = 2$
- $y = \frac{1}{2}(4 + 6) = 5$
- $z = \frac{1}{2}(2 + 3) = 2.5$
- $M (2, 5, 2.5)$

Example

The three points (1, -1, -3), (2, 0, -1) and (a, b, 3) lie on a straight line, find the values of a and b.

- Since $z: -3 < -1 < 3$
- $P_1 (1, -1, -3)$
- $P (2, 0, -1)$
- $P_2 (a, b, 3)$
- $r = \frac{z-z_1}{z_2-z_1} = \frac{-1-(-3)}{3-(-3)} = \frac{1}{3}$
- $x = x_1 + r(x_2 - x_1)$
- $2 = 1 + \frac{1}{3}(a - 1)$
- $a = 4$
- $y = y_1 + r(y_2 - y_1)$

- $0 = -1 + r(b - (-1))$
- $b = 2$

Example

Find the coordinates of the point where the segment joining the points (2, -2, 1) and (5, 1, -2) crosses the xy-plane.

- $z = 0$ (crosses xy-plane)
- $P(x, y, 0) = ?$
- $P_1 (2, -2, 1)$
- $P_2 (5, 1, -2)$
- $r = \frac{z-z_1}{z_2-z_1} = \frac{0-1}{-2-1} = \frac{1}{3}$
- $y = y_1 + r(y_2 - y_1)$
- $y = -2 + \frac{1}{3}(1 - (-2))$
- $y = -1$
- $x = x_1 + r(x_2 - x_1)$
- $x = 2 + \frac{1}{3}(5 - 2)$
- $x = 3$
- $P (3, -1, 0)$

III) The Plane

The Plane

- A flat surface
- **General Form**
 $Ax + By + Cz + D = 0$

- **Intercept Form**

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

- **Three-Points Form**

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- **Expanded form of 3 Points Form**

$$\begin{aligned} & \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix} (x - x_1) \\ & - \begin{vmatrix} x_2 - x_1 & z_2 - z_1 \\ x_3 - x_1 & z_3 - z_1 \end{vmatrix} (y - y_1) \\ & + \begin{vmatrix} y_2 - y_1 & z_2 - z_1 \\ y_3 - y_1 & z_3 - z_1 \end{vmatrix} (z - z_1) = 0 \end{aligned}$$

Two Planes

$$P_1: A_1x + B_1y + C_1z + D_1 = 0$$

$$P_2: A_2x + B_2y + C_2z + D_2 = 0$$

Description of Intersection

Formulas

$P_1 \parallel P_2$	$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
$P_1 \perp P_2$	$A_1A_2 + B_1B_2 + C_1C_2 = 0$
P_1 coincides with P_2	$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$

- Parallel – $P_1 \parallel P_2$
 - $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$
 - $P_1: Ax + By + Cz + D_1 = 0$
 - $P_2: Ax + By + Cz + D_2 = 0$
 - $d = \left| \frac{D_2 - D_1}{\sqrt{A^2 + B^2 + C^2}} \right|$
 - Shortest or Perpendicular distance of the two parallel lines
 - Difference over Pythagorean
- Perpendicular – $P_1 \perp P_2$
 - $A_1A_2 + B_1B_2 + C_1C_2 = 0$
 - SOP of pairs equals zero.
- Acute Angle Between P_1 and P_2
 - $\theta = \arccos \left| \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$
 - Inverse cosine of SOP of pairs over product of Pythagorean.

(Perpendicular or Shortest) Distance from a point

- $P_o (x_o, y_o, z_o)$
- $Ax + By + Cz + D = 0$
- $d = \left| \frac{Ax_o + By_o + Cz_o + D}{\sqrt{A^2 + B^2 + C^2}} \right|$

Volume of Polyhedron Pyramid

- Volume of polyhedron pyramid bounded by plane and the 3-coordinate planes.
- $V_p \left| \frac{D^3}{6ABC} \right| = \left| \frac{abc}{6} \right|$

Example

Find the equation of the plane that passes through (3, -2, 1), (2, 4, -2), and (-1, 3, 2).

- Labeling the points P1, P2, and P3, consecutively...
- $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$
- $$\begin{vmatrix} x - 3 & y + 2 & z - 1 \\ 2 - 3 & 4 + 2 & -2 - 1 \\ -1 - 3 & 3 + 2 & 2 - 1 \end{vmatrix} = 0$$
- $$\begin{vmatrix} x - 3 & y + 2 & z - 1 \\ -1 & 6 & -3 \\ -4 & 5 & 1 \end{vmatrix} = 0$$
- $(6 + 15)(x - 3) - (-1 - 12)(y + 2) + (-5 + 24)(z - 1) = 0$
- $21x - 63 + 13y + 26 + 19z - 19 = 0$
- $21x + 13y + 19z - 56 = 0$**

Example

Find the shortest distance from the point (5, 4, -2) to the plane $21x + 13y + 19z - 56 = 0$.

- Given eq. $\rightarrow Ax + By + Cz + D = 0$
- Given pt. $\rightarrow (x_o, y_o, z_o)$
- $$d = \left| \frac{Ax_o + By_o + Cz_o + D}{\sqrt{A^2 + B^2 + C^2}} \right|$$
- Substitute.
- $d = 2.02 \text{ units}$**

Example

Find the volume of the cube having its 2 faces on the planes $2x - y + 2z - 3 = 0$ and $6x - 3y + 6z + 8 = 0$

- The side of the cube is the distance between the planes.
- The formula for the distance between parallel planes need coefficients A, B, and C to be equal. Thus, we multiply first plane equation mentioned by 3.
- $P_1: 6x - 3y + 6z - 9 = 0$
- $P_2: 6x - 3y + 6z + 8 = 0$
- $$d = \left| \frac{D_2 - D_1}{\sqrt{A^2 + B^2 + C^2}} \right|$$
- Substitute.
- $d = \frac{17}{9}$
- $V = d^3$
- $V = \frac{4913}{729} \text{ cubic units or } 6.74 \text{ cubic units}$**

Example

Find the angle between planes $3x - y + z - 5 = 0$ and $x + 2y + 2z + 2 = 0$

- Labelling the planes as P1 and P2 as mentioned.
- $$\theta = \arccos \left(\frac{A_1 A_2 + B_1 B_2 + C_1 C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \sqrt{A_2^2 + B_2^2 + C_2^2}} \right)$$
- Substituting...
- $\theta = 72.45^\circ$**

Example

Find the volume of the pyramid in the first octant by the plane $6x + 10y + 5z - 30 = 0$ and the 3-coordinate planes.

- Given eq. $\rightarrow Ax + By + Cz + D = 0$
- $$V = \left| \frac{D^3}{6ABC} \right|$$
- Substitute.
- $V = 15 \text{ cubic units}$**

IV) The Line

Line

- Locus of points that can be described as an intersection of 2 planes.

General Form

$$A_1x + B_1y + C_1z + D_1 = 0$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

Two-Points Form

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Symmetric Form (or point and direction nos. form)

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Direction Vector or Nos.

$$DV = [a \ b \ c]$$

Parametric Form

$$x = at + x_1$$

$$y = bt + y_1$$

$$z = ct + z_1$$

- t is called a parametric variable equated to the symmetric form

Example

Given is the line represented by $x - y + 2z = 2$ and $2x + y - z = 1$. Find the equivalent equation in:

- (a) Symmetric Form
- (b) Parametric Form

- Getting two points from the line of intersection...
- Set $x = 0$, and solve for y and z (use systems of equation solver in calculator)
- $P_1 (0,4,3)$
- Set $y = 0$, and solve for x and z
- $P_2 (0.8,0,0.6)$
- Use two-points form:
- $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$
- $\frac{x}{0.8} = \frac{y-4}{-4} = \frac{z-3}{-2.4}$
- Use formula for parametric form:
 - $x = at + x_1$
 - $y = bt + y_1$
 - $z = ct + z_1$
 - OR equate symmetric form to t and solve for x , y , and z .
- $x = 0.8t$
- $y = -4t + 4$
- $z = -2.4t + 3$

Two Lines

$$L_1: \frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

$$L_2: \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

Description of Intersection	Formulas
$L_1 L_2$	$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
$L_1 \perp L_2$	$a_1a_2 + b_1b_2 + c_1c_2 = 0$
L_1 coincides with L_2	$\frac{x_2-x_1}{a_2} = \frac{y_2-y_1}{b_2} = \frac{z_2-z_1}{c_2}$

- Parallel Lines – $L_1 || L_2$
 - $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$
 - Equal ratios of direction nos.

- Perpendicular lines – $L_1 \perp L_2$
 - $a_1a_2 + b_1b_2 + c_1c_2 = 0$
 - Dot product of direction vectors is zero.
 - Product of pairs add up to zero.
- Skew Lines
 - Not parallel, not perpendicular, not intersecting
 - If does not satisfy equations above for parallel and perpendicular lines, then L_1 and L_2 are skew lines.
- Acute Angle Between Lines
 - $\theta = \arccos \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$
 - Inverse cosine of sum of product (SOP) of pairs over product of Pythagorean

Example

Find the acute angle between the lines

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{1} \text{ and } \frac{x}{5} = \frac{y}{4} = \frac{z}{-3}$$

- $[a_1 \ b_1 \ c_1] = [2 \ 2 \ 1]$
- $[a_2 \ b_2 \ c_2] = [5 \ 4 \ -3]$
- $\theta = ?$
- $\theta = \arccos \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$
- $\theta = 45^\circ$

Distance Bet. Skew Lines and Parallel Lines

- Identify direction vectors DV_1 and DV_2
- Get equation of lines in parametric forms.
 - $x_1 = \dots t + \dots$
 - $y_1 = \dots t + \dots$
 - $z_1 = \dots t + \dots$
 - $x_2 = \dots s + \dots$
 - $y_2 = \dots s + \dots$
 - $z_2 = \dots s + \dots$
- Get DV using:
 - $DV = [x_2 - x_1 \ y_2 - y_1 \ z_2 - z_1]$
- Use the equation:
 - $DV_1 \cdot DV = 0$ AND/OR
 - $DV_2 \cdot DV = 0$

- Results to two equations two unknowns s and t.

- Solve for s and t.
- Get magnitude of DV with values of s and t.
 - $d = |DV|$

Example

Find the distance between parallel lines

$$\frac{x}{6} = \frac{y}{-2} = \frac{z}{1}$$

And

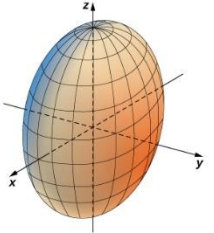
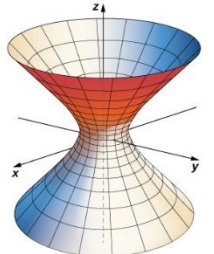
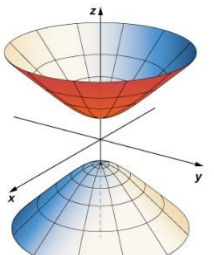
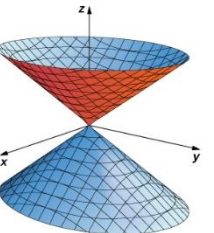
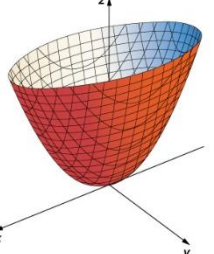
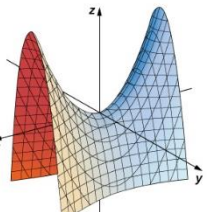
$$\frac{x-7}{6} = \frac{y}{-2} = \frac{z+1}{1}$$

- $DV_1 = [6 \quad -2 \quad 1]$
- $DV_2 = [6 \quad -2 \quad 1]$
- Equate the given eq. to t and s, respectively.
- $x_1 = 6t$
- $y_1 = -2t$
- $z_1 = t$
- $x_2 = 6s + 7$
- $y_2 = -2s$
- $z_2 = s - 1$
- $DV = [x_2 - x_1 \quad y_2 - y_1 \quad z_2 - z_1]$
- $DV = [6s - 6t + 7 \quad -2s + 2t \quad s - t - 1]$
- $DV_1 \cdot DV = 0$ and $DV_2 \cdot DV = 0$
- $6(6s - 6t + 7) - 2(-2s + 2t) + 1(s - t - 1) = 0$
- set $t = 0$
- $s = -1$
- $DV = [1 \quad 2 \quad -2]$
- $d = |DV|$
- $d = \sqrt{1^2 + 2^2 + 2^2}$
- $d = 3 \text{ units}$

V) The Common Quadratic Surface Equations

Common Quadric Surfaces

Characteristics of Common Quadric Surfaces

Ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ <i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: an ellipse In plane $x = r$: an ellipse If $a = b = c$, then this surface is a sphere.	
Hyperboloid of One Sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ <i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the equation for this surface, two of the variables have positive coefficients and one has a negative coefficient. The axis of the surface corresponds to the variable with the negative coefficient.	
Hyperboloid of Two Sheets $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ <i>Traces</i> In plane $z = p$: an ellipse or the empty set (no trace) In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the equation for this surface, two of the variables have negative coefficients and one has a positive coefficient. The axis of the surface corresponds to the variable with a positive coefficient. The surface does not intersect the coordinate plane perpendicular to the axis.	
Elliptic Cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ <i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a hyperbola In plane $x = r$: a hyperbola In the xz-plane: a pair of lines that intersect at the origin In the yz-plane: a pair of lines that intersect at the origin The axis of the surface corresponds to the variable with a negative coefficient. The traces in the coordinate planes parallel to the axis are intersecting lines.	
Elliptic Paraboloid $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ <i>Traces</i> In plane $z = p$: an ellipse In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	
Hyperbolic Paraboloid $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ <i>Traces</i> In plane $z = p$: a hyperbola In plane $y = q$: a parabola In plane $x = r$: a parabola The axis of the surface corresponds to the linear variable.	

Sphere

- General Form
 - $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$
- Center $C(h, k, l)$
 - $h = -\frac{A}{2}$
 - $k = -\frac{B}{2}$
 - $l = -\frac{C}{2}$
 - coordinates =
$$-\frac{\text{coefficient of 1st degree}}{2 \times \text{coefficient of 2nd degree term}}$$
- Standard Form
 - $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
 - $r = \frac{1}{2}\sqrt{A^2 + B^2 + C^2 - 4D}$
 - $r = \sqrt{\left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2 + \left(\frac{C}{2}\right)^2 - D}$
- For vertex at origin:
 - $x^2 + y^2 + z^2 = r^2$

Example

Given is equation of sphere

$$x^2 + y^2 + z^2 = 6x + 8z$$

Find the:

- Location of center
- Radius

- $x^2 - 6x + y^2 + z^2 - 8z = 0$
- coordinates =
$$-\frac{\text{coefficient of 1st degree}}{2 \times \text{coefficient of 2nd degree term}}$$
- $h = -\frac{-6}{2 \times 1} = 3$
- $k = -\frac{0}{2 \times 1} = 0$
- $l = -\frac{-8}{2 \times 1} = 4$
- **C (3, 0, 4)**
- A B C are coefficient of 1st degree terms
- $r = \sqrt{\left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2 + \left(\frac{C}{2}\right)^2 - D}$
- **r = 5**

Ellipsoid + Properties

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

(Vertex at 0)

- $V = \frac{4\pi}{3}abc$
- Largest rectangular parallelepiped inscribed in ellipsoid
 - $L = \frac{2a}{\sqrt{3}}; W = \frac{2b}{\sqrt{3}}; H = \frac{2c}{\sqrt{3}}$
 - $V = LWH = \frac{8abc}{3\sqrt{3}}$

Example

Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $x^2 + 3y^2 + 9z^2 = 9$

- Divide by 9.
- $\frac{x^2}{3^2} + \frac{y^2}{3} + \frac{z^2}{1^2} = 1$
- $a = 3$
- $b = \sqrt{3}$
- $c = 1$
- $V = \frac{8}{3\sqrt{3}}abc$
- **V = 8 cubic units**

Hyperboloid

- $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
 - Axis on z-axis
 - Hyperboloid with one sheet or nappe
- $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$
 - Axis on x-axis
 - Hyperboloid with two sheets or two napes
- Refer to pictures before this page for determining axis and number of sheets.
 - Number of negative signs is number of sheets.
 - Axis on the one different sign (yung nag-iisang positive or nag-iisang negative)

The Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \text{ or } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

- Axis on z-axis.
- Determining axis:
 - Axis is on isolated variable in one side.
 - Axis is on the one negative signed variable.

Elliptic Paraboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

- Level curves are ellipses
 - Curves at different values of z.
- Axis on isolated variable without square (z-axis in this case).

Hyperbolic Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$$

- Level curves are hyperbolas
 - Curves at different values of z.
- Axis on isolated variable without square (z-axis in this case).

Cylinder

- Hyperbolic Cylinder
 - $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Elliptic Cylinder
 - $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
 - Circular Cylinder
 - $a = b$
- Axis on missing variable (z-axis in this case)
- Level curves are hyperbolas, ellipses, or circles, that are the same in all values of z.

Parabolic Cylinder

- $\frac{x^2}{a^2} = \frac{y}{b}$
- Axis on missing variable (z-axis in this case)
- Level curves are parabolas that is the same in all values of z.

VI) Equation of Surface of Revolution

Equation of Surface of Revolution

- The equation of curve will be given
- The axis about which the curve will be revolved will be given

Curve Revolved About What Axis?	Replace What?	With What?
x-axis	y or z	$\sqrt{y^2 + z^2}$
y-axis	x or z	$\sqrt{x^2 + z^2}$
z-axis	x or y	$\sqrt{x^2 + y^2}$
	The variable that is not the axis.	Pythagorean theorem of variables that are not the axis.

Example

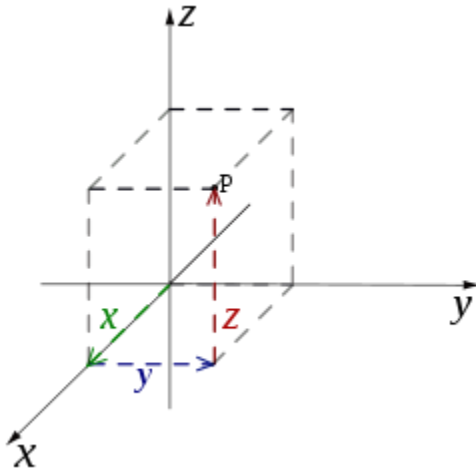
Find the equation of the surface generated by revolving the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the x-axis.

- Replace y or z with $\sqrt{y^2 + z^2}$
- $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$
- $a > b$
- One big radius, two small radii...
- Shape is prolate like rugby ball.

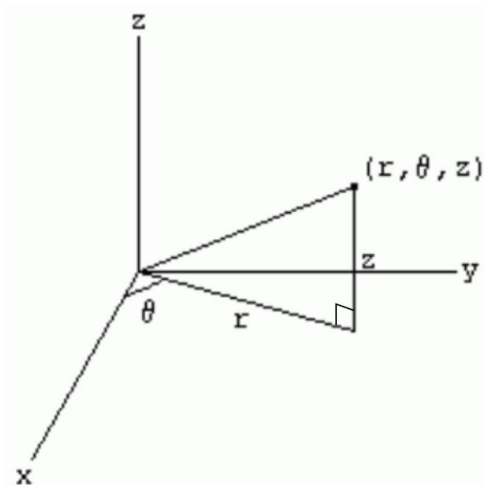
VII) Transformation of Coordinate System

Coordinate Systems

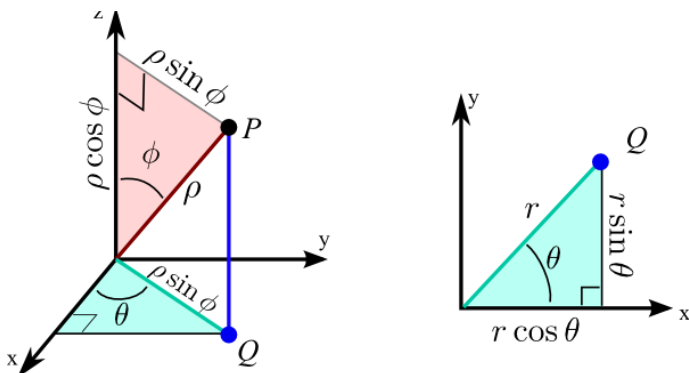
- RCS – Rectangular Coordinate System



- CCS – Cylindrical Coordinate System



- SCS – Spherical Coordinate System



To \ From	RCS	CCS	SCS
RCS	---	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $z = z$	$\rho = \sqrt{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$
CCS	$x = r \cos \theta$ $y = r \sin \theta$ $z = z$	---	$\rho = \sqrt{r^2 + z^2}$ $\theta = \theta$ $\phi = \cos^{-1}\left(\frac{z}{\rho}\right)$
SCS	$x = \rho \sin \phi \cos \theta$ $y = \rho \sin \phi \sin \theta$ $z = \rho \cos \phi$	$r = \rho \sin \phi$ $\theta = \theta$ $z = \rho \cos \phi$	---

Example

Find the equivalent equation of the $2x^2 + 2y^2 + z^2 = 15$ in cylindrical coordinate system.

- $x = r \cos \theta$
- $y = r \sin \theta$
- $z = z$
- $2r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + z^2 = 15$
- $2r^2 + z^2 = 15$

Example

Find the equivalent equation of $\phi = 60^\circ$ in spherical coordinates system to the rectangular coordinates system.

- $\phi = \arccos\left(\frac{z}{\rho}\right)$
- $\cos \phi = \frac{z}{\rho}$
- $\cos 60^\circ = \frac{z}{\rho}$
- $\frac{1}{2} = \frac{z}{\rho}$
- $\rho = \sqrt{x^2 + y^2 + z^2}$
- $\frac{1}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$
- $\sqrt{x^2 + y^2 + z^2} = 2z$
- $x^2 + y^2 + z^2 = 4z^2$
- $x^2 + y^2 = 3z^2$

DIFFERENTIAL CALCULUS

differential calculus

I. Derivative

Derivative

- Let $y = f(x)$
 - Then derivative of y with respect to x is:
 - $\frac{dy}{dx}$
 - y'
 - $f'(x)$
- Importance of derivative
 - $\frac{dy}{dx} = m$
 - Derivative tells slope of curve or tangent line to curve.
 - $\frac{dy}{dx} = r$
 - Derivative is the rate of change of a dependent quantity y over an independent quantity x
 - Ex:
 - $v = \frac{ds}{dt}$
 - Instantaneous velocity
 - $a = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{ds}{dt} \right) = \frac{d^2s}{dt^2}$
 - Instantaneous acceleration
 - First derivative of velocity
 - Second derivative of position or distance

II. Evaluating Derivatives

Evaluation of Derivatives

- For explicit functions $y = f(x)$
 - Use the formulas.
- For implicit functions $f(x, y) = 0$
 - Derive both sides of the equation by x .
 - Use chain rule for the variable y .

Ex: $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx} = 2y y'$

- Solve for y' or $\frac{dy}{dx}$
- Derivative of implicit functions shortcut
 - Model: $f = g(x, y) = 0$
 - $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$
- For parametric equations
 - Model:
 - $x = f(t)$
 - $y = g(t)$
 - Use $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Example

Find the derivative of $y = \sin^2 x$

- $y' = 2 \sin x \cdot \frac{d}{dx}(\sin x)$ (Chain Rule)
- $y' = 2 \sin x \cdot \cos x$
- $y' = \sin 2x$ (From Identity)

Example

Find the derivative of y implicitly defined by the equation: $x^2y + xy^2 - 2x + 3y - 5 = 0$.

- $f = x^2y + xy^2 - 2x + 3y - 5$
- $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$
- $\frac{\partial f}{\partial x} = 2xy + y^2 - 2$
- $\frac{\partial f}{\partial y} = x^2 + 2xy + 3$
- $\frac{dy}{dx} = -\frac{2xy + y^2 - 2}{x^2 + 2xy + 3}$

Example

Find $\frac{dy}{dx}$ if y is defined by the parametric equations:

$$x = a \cos^3 t$$

$$y = a \sin^3 t$$

(Parametric Equations of Astroid or 4-cusps hypocycloid)

- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- $\frac{dx}{dt} = 3a \cos^2 t (-\sin t)$
- $\frac{dx}{dt} = -3a \sin t \cos^2 t$
- $\frac{dy}{dt} = 3a \sin^2 t (\cos t)$
- $\frac{dy}{dt} = 3a \sin^2 t \cos t$
- $\frac{dy}{dx} = \frac{3a \sin^2 t \cos t}{-3a \sin t \cos^2 t}$
- $\frac{dy}{dx} = -\frac{\sin t}{\cos t}$
- $\frac{dy}{dx} = -\tan t$

A) Slope of Line Tangent to the Curve

Instantaneous Slope

- Rectangular Coordinate System: $m = \frac{dy}{dx}$
- Polar Coordinates System:
 - Model:
 - $r = f(\theta)$
 - $x = r \cos \theta$
 - $y = r \sin \theta$
 - $m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$
 - $m = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$

Example

Find the slope of the parabola $y^2 = 4x$ at the point (1,2).

- $m = \frac{dy}{dx}$
- $f = y^2 - 4x$
- $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$
- $\frac{dy}{dx} = -\frac{-4}{2y} = \frac{4}{2y}$
- $m|_{(1,2)} = \frac{dy}{dx}|_{(1,2)}$
- $m|_{(1,2)} = \frac{4}{2(2)} = 1$
- **$m = 1$**

Example

Find the slope of the cardioid:

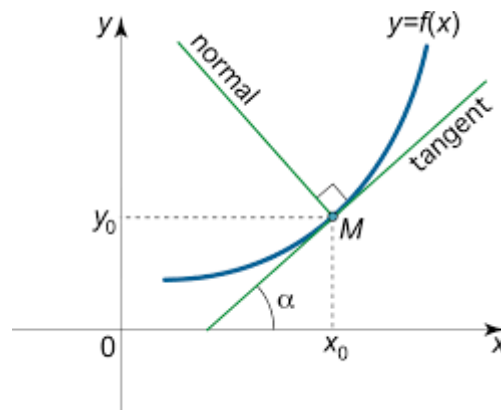
$$r = 2(1 + \cos \theta)$$

At the point where $\theta = \frac{\pi}{6}$

- $x = r \cos \theta$
- $y = r \sin \theta$
- $m = \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$
- $m = \frac{\frac{d}{d\theta}(r \sin \theta)}{\frac{d}{d\theta}(r \cos \theta)}$
- $m = \frac{\frac{d}{d\theta}(2(1 + \cos \theta) \sin \theta)}{\frac{d}{d\theta}(2(1 + \cos \theta) \cos \theta)}$
- Evaluate derivatives at $\theta = \frac{\pi}{6}$ in calculator.
- **$m = -1$**

B) Equation of Tangent Line and Normal Line

Tangent Line and Normal Line



- Tangent Line: $y - y_0 = m(x - x_0)$
 - Slope of line same as slope of curve at point of intersection.
 - $m_{TL} = m = \frac{dy}{dx}$
- Normal Line: $y - y_0 = -\frac{1}{m}(x - x_0)$
 - Slope of line is negative reciprocal of slope of curve at point of intersection.
 - $m_{NL} = -\frac{1}{m_{TL}} = -\frac{1}{m}$

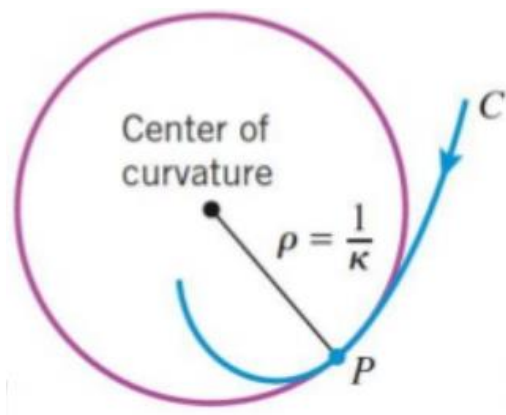
Example

Find the equation of the tangent line and normal line to the parabola $y^2 = 4x$ at the point $(1,2)$.

- $m = \frac{dy}{dx}$
- $2y \frac{dy}{dx} = 4$
- $\frac{dy}{dx} = \frac{4}{2y} = \frac{4}{2(2)} = 1$
- $m_T = m = 1$
- $m_N = -\frac{1}{m} = -1$
- $y - y_o = m(x - x_o)$
- Tangent Line
 - $y - 2 = x - 1$
 - $x - y + 1 = 0$
- Normal Line
 - $y - 2 = -(x - 1)$
 - $x + y - 3 = 0$

C) Curvature and Radius of Curvature

Curvature and Radius of Curvature



- Rectangular CS: $\rho = \frac{[1+(y')^2]^{\frac{3}{2}}}{y''}$
 - $s = \rho\alpha$
 - Arc length = radius of curvature times angle
 - $ds = \rho d\alpha$
 - $\rho = \frac{ds}{d\alpha}$
 - $m = y' = \tan \alpha$
 - $\alpha = \arctan(y')$
 - $d\alpha = \frac{y'' dx}{1+(y')^2}$
 - $ds = \sqrt{1 + (y')^2} dx$

Arc length formula

- $\rho = \frac{\sqrt{1+(y')^2} dx}{\frac{y'' dx}{1+(y')^2}}$
- $\rho = \frac{[1+(y')^2]^{\frac{3}{2}}}{y''}$
- Polar CS: $\rho = \frac{[r^2+(r')^2]^{\frac{3}{2}}}{r^2+2(r')^2-r r''}$
 - From RCS Formula, derivative equation for PCS can be used to substitute for y' .
- Curvature, κ
 - $\kappa = \frac{1}{\rho}$
- Sign
 - Positive curvature or radius means positive concavity.
 - Negative curvature or radius means negative concavity.

Example

Find the radius of curvature of the sinusoidal curve $y = \sin x$ at $x = \frac{\pi}{2}$.

- $\rho = \frac{[1+(y')^2]^{\frac{3}{2}}}{y''}$
- $y = \sin x$
- $y' = \cos x = \cos \frac{\pi}{2} = 0$
- $y'' = -\sin x = -\sin \frac{\pi}{2} = -1$
- $\rho = \frac{[1+0^2]^{\frac{3}{2}}}{-1}$
- $\rho = -1$

Example

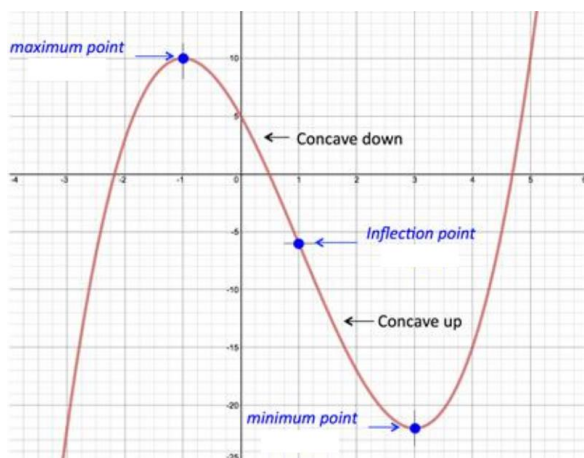
Find the curvature of the lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$ at $\theta = 0$.

- $\rho = \frac{[r^2+(r')^2]^{\frac{3}{2}}}{r^2+2(r')^2-r r''}$
- $r^2 = a^2 \cos 2\theta$
- Subs. $\theta = 0$
- $\underline{r = a}$
- $2r r' = -2a^2 \sin 2\theta$
- $r r' = -a^2 \sin 2\theta$
- Subs $r = a$ and $\theta = 0$
- $\underline{r' = 0}$

- $r r'' + (r')^2 = -2a^2 \cos 2\theta$
- Substitute $r = a$, $r' = 0$, and $\theta = 0$
- $ar'' = -2a^2$
- $r'' = -2a$
- $\rho = \frac{[a^2 + 0^2]^{\frac{3}{2}}}{a^2 + 2(0)^2 - a(-2a)}$
- $\rho = \frac{a^3}{3a^2}$
- $\rho = \frac{a}{3}$
- $\kappa = \frac{1}{\rho}$
- $\kappa = \frac{3}{a}$

D) Maximum and Minimum Point, and Point of Inflection

Definition of Points



- Maximum and Minimum Points are called critical points.
- Critical Point: $\frac{dy}{dx} = 0$
- Concave Down: $\frac{d^2y}{dx^2} < 0$
- Concave Up: $\frac{d^2y}{dx^2} > 0$
- Point of Inflection: $\frac{d^2y}{dx^2} = 0$
 - Point at which curve changes concavity
- Maximum point: $\frac{d^2y}{dx^2} < 0$ and $\frac{dy}{dx} = 0$
 - Concave down and critical point.
- Minimum point: $\frac{d^2y}{dx^2} > 0$ and $\frac{dy}{dx} = 0$
 - Concave up and critical point.

Example

Find the maximum point, minimum point, and point of inflection for the polynomial curve:

$$y = x^3 + 2x^2 - 4x + 5$$

- $y' = 3x^2 + 4x - 4 = 0$
- Critical points at $x = \frac{2}{3}$ and $x = -2$
- $y'' = 6x + 4 = 0$
- Inflection point at $x = -\frac{2}{3}$
- Make a table

	y''	Remarks
$x < -\frac{2}{3}$	Negative	Concave Down Has maximum
$x = -\frac{2}{3}$	0	Inflection Point
$x > -\frac{2}{3}$	Positive	Concave Up Has minimum

- Maximum point at $x = -2$
- Inflection point at $x = -\frac{2}{3}$
- Minimum point at $x = \frac{2}{3}$
- Substituting to $y = x^3 + 2x^2 - 4x + 5$
- Max. point $(-2, 13)$
- Inflection point $(-\frac{2}{3}, \frac{223}{27})$
- Min. point $(\frac{2}{3}, \frac{95}{27})$

E) Maxima Minima Problems

Steps for Solution

- Identify quantity to maximize or minimize.
 - $Q_{min} = ?$
- Relate Q to other variables by use of theorem or formulas, or as stated in the problem.
 - $Q = f(x, y)$
- Express Q in terms of one variable.
 - $Q = f(x)$
- Derive and equate to zero (get critical point).
 - $Q' = f'(x) = 0 \rightarrow x = a$
- Use value of x to solve the requirements of the problem.
 - $Q_{min} = f(a)$

Example

Find 2 numbers with sum equal to 36 so that the product of the 1st no. with the square of the other is a maximum.

- $x = ?$
- $y = ?$
- $x + y = 36$
- $P = xy^2 \rightarrow \text{maximize}$
- $x = 36 - y$
- $P = (36 - y)(y^2)$
- $P = 36y^2 - y^3$
- $P' = 72y - 3y^2 = 0$
- Critical points at $y = 0, y = 24$
- Reject $y = 0$
- For $y = 0, P = 0$
- For $y = 24, P = 6912$
- P is maximum when $y = 24$
- $x = 36 - y = 12$
- $x = 12$
- $y = 24$

Given sum, Maximize Product Problem Shortcut

$$P = x^m y^n$$

$$x + y = k$$

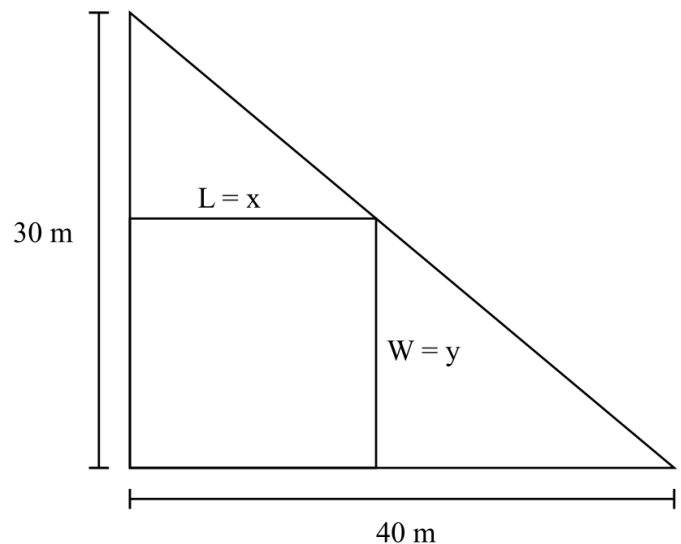
$$x = \frac{mk}{m+n}$$

$$y = k - x$$

- Using shortcut for previous example:
 - $m = 1$
 - $n = 2$
 - $k = 36$
 - $x = \frac{36}{1+2} = 12$
 - $y = 36 - 12 = 24$

Example

A triangular lot is bounded by 2 perpendicular streets on 2 sides which are 30 m and 40 m long, respectively. Find the dimensions and base area of the largest rectangular building that can be constructed with 2 sides facing the streets.



- $A_B \rightarrow \text{maximize}$
- $L \times W = ?$
- Putting the right angled point at the origin, the hypotenuse is a line with intercepts $a = 40$ and $b = 30$
- $\frac{x}{40} + \frac{y}{30} = 1$
- $3x + 4y = 120$
- $4y = 120 - 3x$
- $y = 30 - \frac{3}{4}x$
- $A_B = LW = xy = x\left(30 - \frac{3}{4}x\right)$
- $A_B = 30x - \frac{3}{4}x^2$

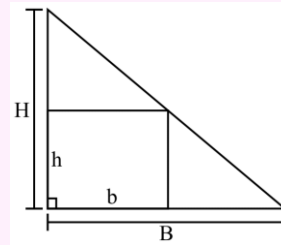
- $A'_B = 30 - \frac{3}{2}x = 0$
- Critical point at $x = 20$
- $y = 30 - \frac{3}{4}(20)$
- $y = 15$
- $A_{B,max} = 15 \times 20$
- $L \times W = 15 \text{ by } 20$
- $A_{B,max} = 300 \text{ m}^2$

Largest Inscribed Rectangle Shortcut

Inscribed in...

Formulas

A right triangle

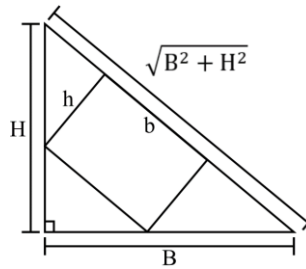


$$h = \frac{1}{2}H$$

$$b = \frac{1}{2}B$$

$$a = \frac{1}{4}BH$$

$$a = \frac{1}{2}A = \frac{1}{2}\left(\frac{1}{2}BH\right)$$

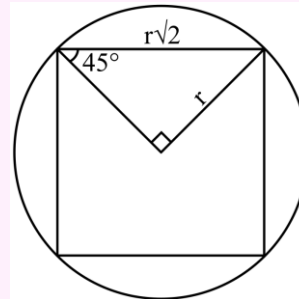


$$b = \frac{1}{2}\sqrt{B^2 + H^2}$$

$$h = \frac{1}{2}\frac{BH}{\sqrt{B^2 + H^2}}$$

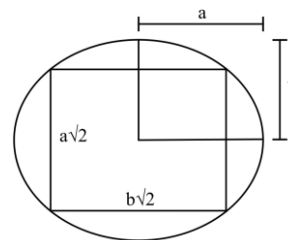
$$a = \frac{1}{2}A = \frac{1}{4}BH$$

$$h = \frac{a}{b} = \frac{\frac{1}{2}\left(\frac{1}{2}BH\right)}{\frac{1}{2}\sqrt{B^2 + H^2}}$$



$$s = r\sqrt{2}$$

$$A = 2r^2$$



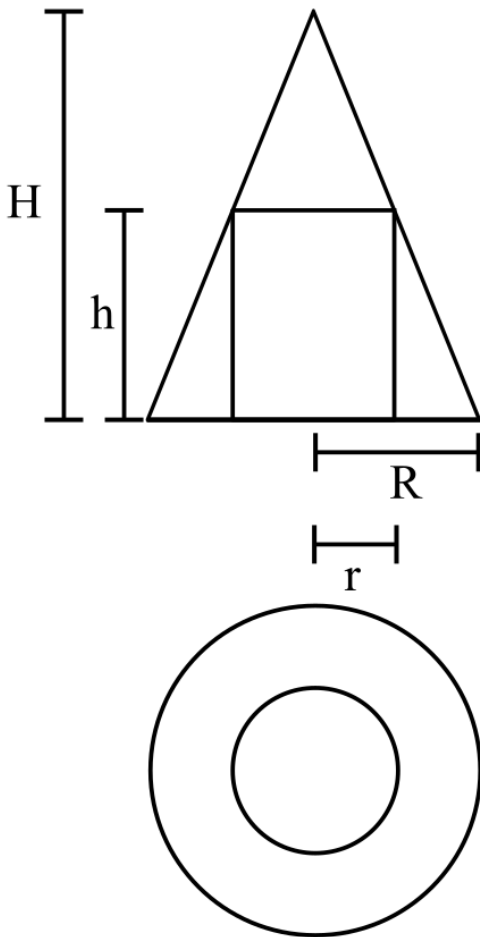
$$L = a\sqrt{2}$$

$$W = b\sqrt{2}$$

$$A = 2ab$$

Example

Find the dimensions and volume of the largest cylinder that can be cut from a given cone with radius 3 m and height 6 m.



- $H = 3 \text{ cm}$
- $R = 6 \text{ cm}$
- $r = ?$
- $h = ?$
- $V_{max} = ?$
- Putting base center of triangle in figure at origin. The slant line equation is
- $\frac{x}{R} + \frac{y}{H} = 1$
- $\frac{x}{3} + \frac{y}{6} = 1$
- $2x + y = 6$
- $y = 6 - 2x$
- $V = \pi r^2 h$
- $V = \pi x^2 y$
- $V = \pi x^2 (6 - 2x)$
- $V = \pi (6x^2 - 2x^3)$
- $V' = \pi (12x - 6x^2) = 0$

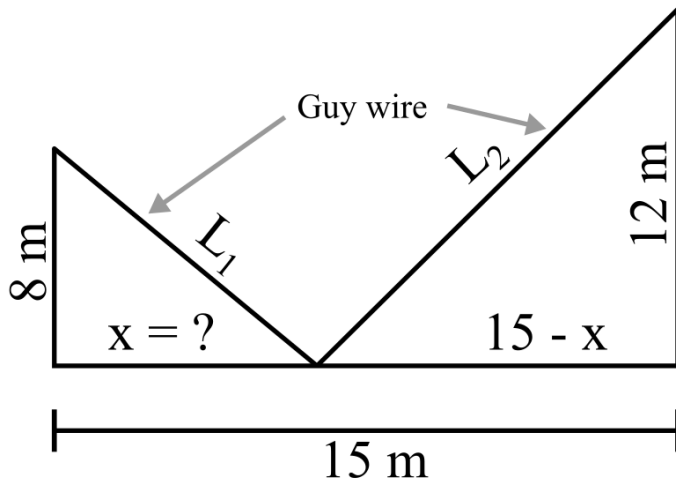
- Critical point at $x = 0$ and $x = 2$
- Reject radius of 0.
- $y = 6 - 2(2) = 2$
- $r = 2 \text{ m}$
- $h = 2 \text{ m}$
- $V = \pi r^2 h =$
- $V_{max} = \pi (2)^2 (2)$
- $V_{max} = 8\pi \text{ m}^3$

Largest Inscribed Cylinder Shortcut

Inscribed in a...	Formulas
Cone 	$r = \frac{2}{3}R$ $h = \frac{1}{3}H$ $v = \frac{4}{9}V = \frac{4}{9}\left(\frac{1}{3}\pi R^2 H\right)$ $v = \pi r^2 h$
Sphere 	$d = \sqrt{\frac{2}{3}}D$ $h = \frac{1}{\sqrt{3}}D$ $v = \frac{V}{\sqrt{3}} = \frac{1}{\sqrt{3}}\left(\frac{4}{3}\pi\left(\frac{D}{2}\right)^3\right)$ $V = \pi\left(\frac{d}{2}\right)^2 h$

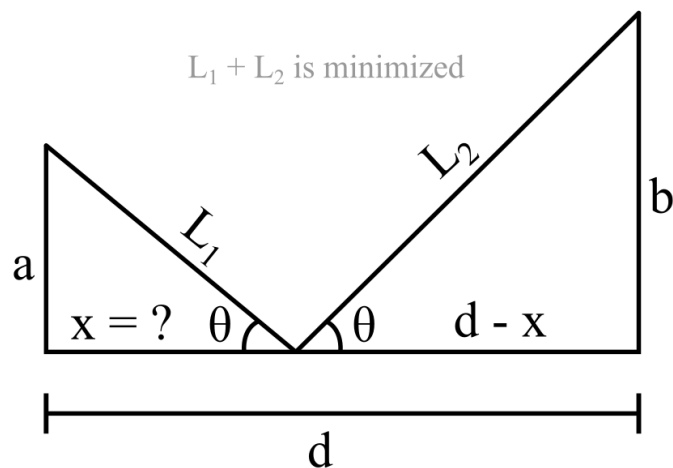
Example

Given are 2 posts which are 15 m apart and 8 m and 12 m tall. Find the distance of common stake between them from 8 m post for minimum length of guy wire running from the top of the posts to the stake.



- $L = L_1 + L_2$
- $L \rightarrow \text{minimize}$
- $L = \sqrt{x^2 + 8^2} + \sqrt{(x - 15)^2 + 12^2}$
- $L = \sqrt{x^2 + 64} + \sqrt{x^2 - 30x + 369}$
- $L' = \frac{2x}{2\sqrt{x^2+64}} + \frac{2x-30}{2\sqrt{x^2-30x+369}} = 0$
- Critical point at $x = 6$
- $x = 6 \text{ m}$

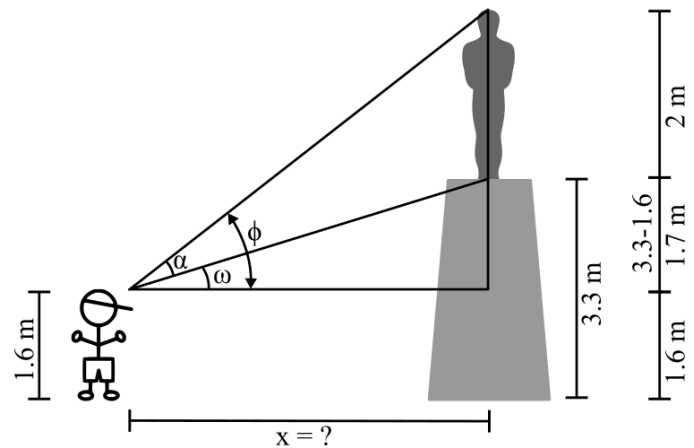
Minimize Length Problem Shortcut



- $\tan \theta = \frac{a}{x} = \frac{b}{d-x}$
- $x = d \left(\frac{a}{a+b} \right)$
 - Parang voltage division lang...

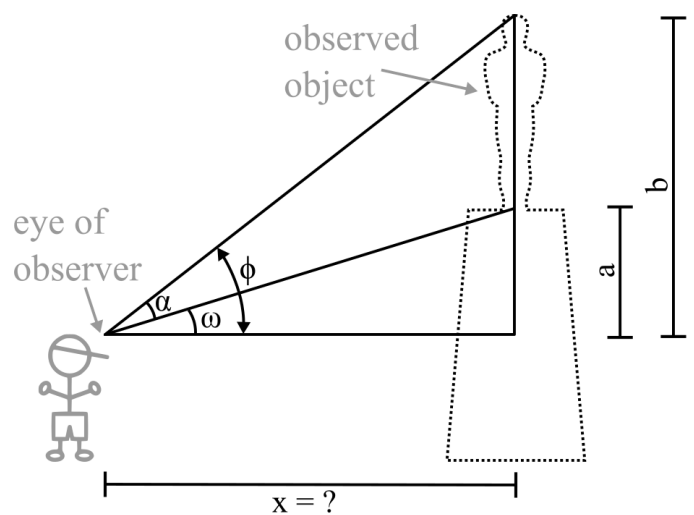
Example

A boy 1.6 m tall is viewing a 2 m tall statue atop a 3.3 m tall pedestal. How far from the bases of the pedestal must the boy stand on level ground to get the best view of the statue



- $\alpha \rightarrow \text{maximize}$
- $\alpha = \phi - \omega$
- $\tan \phi = \frac{2+1.7}{x} = \frac{3.7}{x}$
- $\tan \omega = \frac{1.7}{x}$
- $\alpha = \arctan \frac{3.7}{x} - \arctan \frac{1.7}{x}$
- $\alpha' = \frac{-\frac{3.7}{x^2}}{1 + \left(\frac{3.7}{x}\right)^2} - \frac{-\frac{1.7}{x^2}}{1 + \left(\frac{1.7}{x}\right)^2} = 0$
- $\alpha' = -\frac{3.7}{x^2 + 3.7^2} + \frac{1.7}{x^2 + 1.7^2} = 0$
- Critical point at $x = 2.51$
- $x = 2.51 \text{ m}$

Best View Shortcut



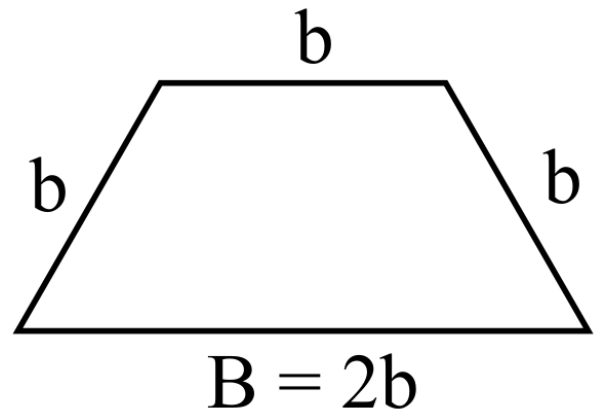
- $x = \sqrt{ab}$

Example

A rectangular lot is bounded by 2 perpendicular walls. Find the area of the largest lot if the fencing material is 20m.

- $A \rightarrow$ maximize
- $b + h = 20$
- $h = 20 - b$
- $A = bh$
- $A = b(20 - b)$
- $A = 20b - b^2$
- $A' = 20 - 2b = 0$
- Critical point at $b = 10$
- $h = 20 - 10 = 10$
- $A = bh = 10 \cdot 10$
- $A = 100 \text{ m}^2$

Max. Area for Trapezoid w/ 3 Sides Equal Shortcut

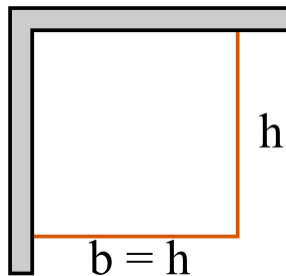


- $B = 2b$
- $A = 3(\frac{1}{2}b^2 \sin 60^\circ)$
- Shape should be half a regular hexagon.

Fencing Material for Maximum Area Shortcut

Scenario

Bounded by two walls



Formulas

$$b + h = F$$

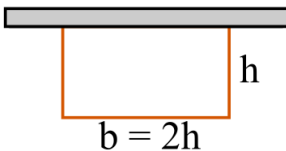
$F =$ fencing material

$$b = h = \frac{L}{2}$$

$$A = b^2 = \left(\frac{L}{2}\right)^2$$

Shape should be a square.

Bounded by one wall



$$b + 2h = F$$

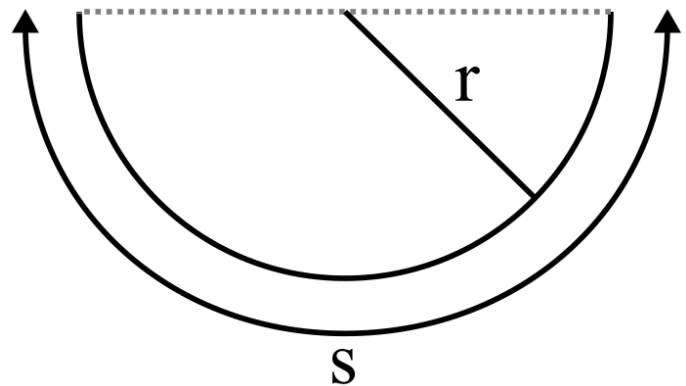
$$b = 2h$$

$$b = \frac{F}{2}$$

$$h = \frac{F}{4}$$

$$A = bh = \frac{F^2}{8}$$

Max. Discharge for Canal



- Max. Discharge means max. area
- Shape should be a semi-circle
- $s = \pi r$
- $r = \frac{s}{\pi}$

Example

The three sides of a trapezoid are each 10 cm long, how long must the fourth side be if the area is a maximum?

- $b = 10 \text{ cm}$
- $B = 2b$
- $B = 20 \text{ cm}$

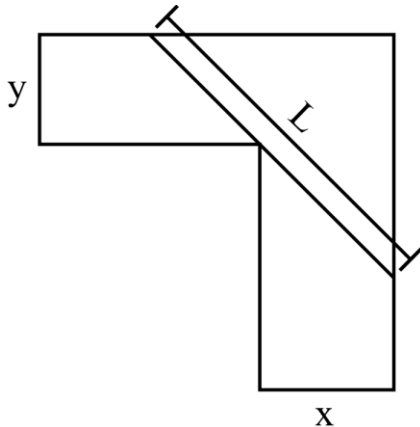
Example

An open cylindrical trough is constructed by bending a given sheet of tin and breadth $2a$. Find the radius of the cylinder of which are through forms when the capacity of the trough is a maximum.

- $s = 2a$
- $s = \pi r$
- $r = \frac{s}{\pi}$
- $r = \frac{2a}{\pi}$
- $r = 0.637\pi$

Example

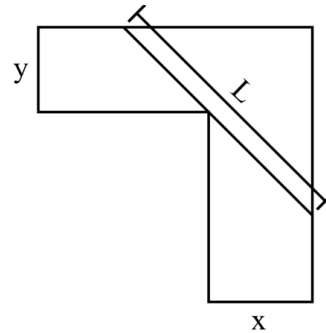
A steel girder 8 m long is moved on rollers along a passageway 4 m wide and into a corridor at right angles to the passageway. Neglecting the width of the girder, how wide must the corridor be?



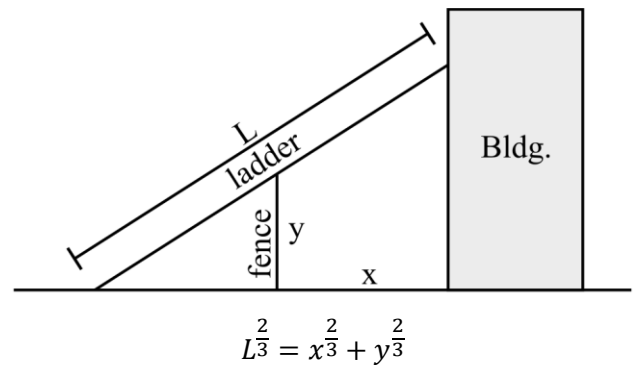
- $L^{\frac{2}{3}} = x^{\frac{2}{3}} + y^{\frac{2}{3}}$
- $8^{\frac{2}{3}} = 4^{\frac{2}{3}} + b^{\frac{2}{3}}$
- $b = 1.8 \text{ m}$

Passage Way or Ladder Shortcut

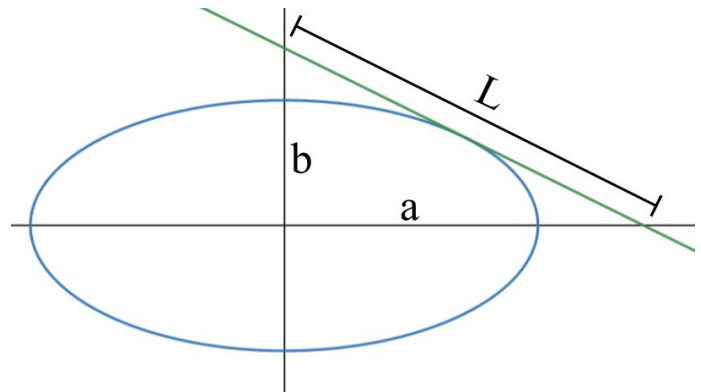
(Long object made to pass across two perpendicular passages, minimum width of passage or longest ladder)



(Shortest Ladder That can reach the building)



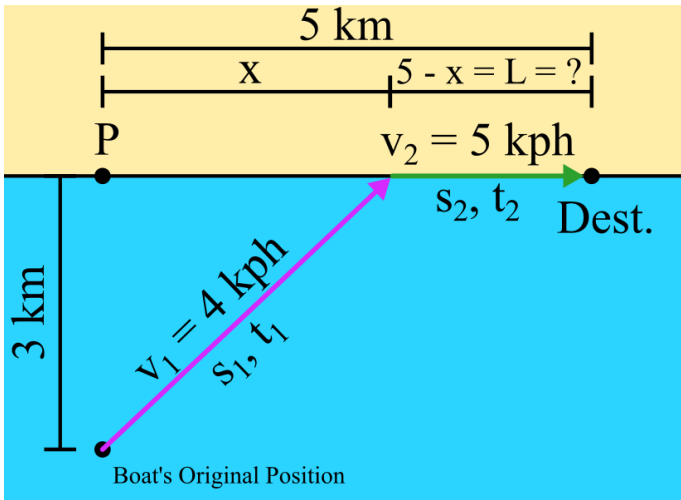
Shortest Tangent Line to Ellipse Shortcut



- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- $L = a + b$
 - Shortest Line tangent to ellipse and meeting or ending at the coordinate axes.

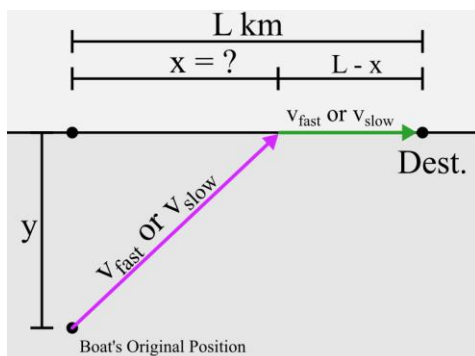
Example

A person in a rowboat is 3 km from a point P on a straight shore while his destination is 5 km directly east of point P. If he is able to row 4 kph and walk 5 kph, how far from the destination must he land on the shore in order to reach his destination in the shortest possible time.



- $t = t_1 + t_2 \rightarrow \text{minimize}$
- $v = \frac{s}{t} \rightarrow t = \frac{s}{v}$
- $t = \frac{s_1}{v_1} + \frac{s_2}{v_2}$
- $t = \frac{\sqrt{x^2 + 3^2}}{4} + \frac{5-x}{5}$
- $t' = \frac{2x}{4(2\sqrt{x^2 + 3^2})} - \frac{1}{5} = 0$
- Critical point at $x = 4$
- $L = 5 - x = 5 - 4 = 1$
- **$L = 1 \text{ km}$**

Slant Travel Then Straight Travel Shortcut



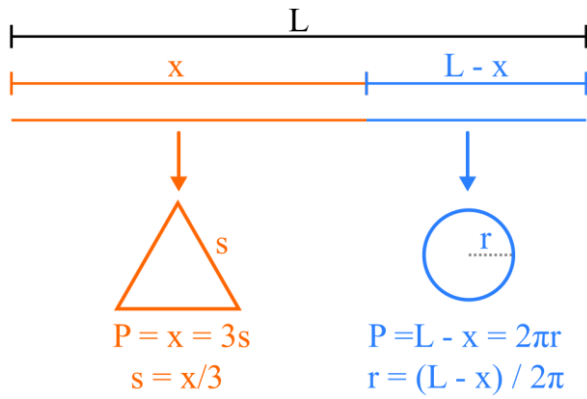
- $x = \frac{v_{\text{slow}} \cdot y}{\sqrt{(v_{\text{fast}})^2 - (v_{\text{slow}})^2}}$
- Solving previous problem:
 - $x = \frac{4 \cdot 3}{\sqrt{5^2 - 4^2}} = 4$
 - **$L - x = 5 - 4 = 1 \text{ km}$**

Example

A piece of wire 36 cm long is cut in two, one part being bent in the shape of an equilateral triangle and the other in the form of a circle. Find the length of wire for the equilateral triangle if the sum of the two areas of these two figures is to be minimum.

- $A_T \rightarrow \text{minimize}$
- $A_{\text{triangle}} + A_{\text{circle}} = A_T$
- $x + y = 36$ (36 cm long is cut in two)
- $s = \frac{x}{3}$ (x is for equilateral triangle, s is side of triangle)
- $y = 2\pi r$ (y is for circle that is the circumference, r is radius of circle)
- $x = ?$
- $y = L - x$
- $L - x = 2\pi r$
- $r = \frac{L-x}{2\pi}$
- $A_{\text{triangle}} + A_{\text{circle}} = A_T$
- $A_T = \frac{1}{2}s^2 \sin 60^\circ + \pi r^2$
- $A_T = \frac{\sqrt{3}}{4}s^2 + \pi r^2$
- $A_T = \frac{\sqrt{3}}{4}\left(\frac{x}{3}\right)^2 + \pi\left(\frac{L-x}{2\pi}\right)^2$
- $A'_T = \frac{\sqrt{3}}{4 \cdot 9} \cdot 2x + \frac{1}{4\pi} 2(x - L) = 0$
- $\frac{\sqrt{3}}{18} 2x + \frac{1}{2\pi} (x - L) = 0$
- $\frac{\sqrt{3}}{9} x + \frac{1}{\pi} (x - L) = 0$
- $\left(\frac{\sqrt{3}}{9} + \frac{1}{\pi}\right) x = \frac{L}{\pi}$
- $\left(\frac{\pi\sqrt{3}+9}{9\pi}\right) x = \frac{L}{\pi}$
- $x = \frac{9L}{\pi\sqrt{3}+9}; L = 36$
- Critical point at $x = 22.4355$
- **$x = 22.44 \text{ cm}$**

Wire to Equilateral and Circle Shortcut



- $x = \frac{9L}{\pi\sqrt{3}+9}$

Example

In how many equal parts can a wire 50 cm long be cut so that the product of its parts is a maximum?

- $L = 50 \text{ cm}$
- $P = \text{product} = \left(\frac{L}{x}\right)^x$
- $\ln P = x \ln \left(\frac{L}{x}\right)$
- $\frac{1}{P} P' = x \frac{x}{L} \left(-\frac{L}{x^2}\right) + \ln \left(\frac{L}{x}\right)$
- $P' = P \left(-1 + \ln \left(\frac{L}{x}\right)\right) = 0$
- $\ln \left(\frac{L}{x}\right) = 1$
- $\frac{L}{x} = e$
- $x = \frac{L}{e}$
- $x = \frac{50}{e} = 18.39$
- $P(18) = \left(\frac{50}{18}\right)^{18} = 96.95M$
- $P(19) = \left(\frac{50}{19}\right)^{19} = 96.41M$
- $x = 18 \text{ cm}$

Maximize Product of Equal Parts Shortcut

- N or L = given number or length of wire
- x = number of divisions or equal pieces cut out
- P = product of parts maximized
- $P = \left(\frac{L}{x}\right)^x = \left(\frac{N}{x}\right)^x$
- $x = \frac{L}{e} = \frac{N}{e}$

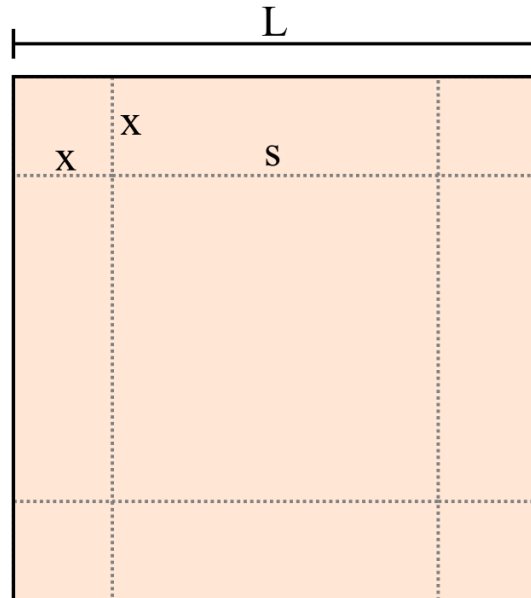
Example

Find the area of the largest rectangle that can be inscribed in an ellipse $x^2 + 4y^2 = 4$ with the sides parallel to the axes of the ellipse.

- $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$
- $a = 2; b = 1$
- $W = a\sqrt{2}$
- $H = b\sqrt{2}$
- $A = WH = 2ab$
- $A = 2 \cdot 2 \cdot 1$
- $A = 4 \text{ units}$

Example

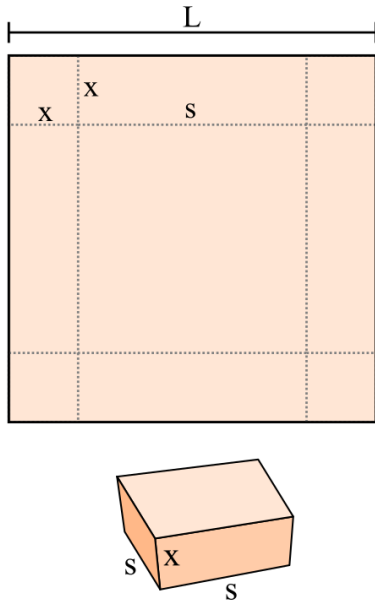
A cardboard 20 in x 20 in is to be formed into a box by cutting four equal squares and folding the edges. Find the volume of the largest box.



- $V = s^2 x$
- $s + 2x = L$
- $s = L - 2x$
- $V = (2x - L)^2 x$
- $V = 4x^3 - 4Lx^2 + L^2 x$
- $V' = 12x^2 - 8Lx + L^2 = 0$
- $x = \frac{4L \pm \sqrt{16L^2 - 12L^2}}{12} = \frac{4L \pm 2L}{12}$
- $x = \frac{L}{6}, \frac{L}{2}$
 - Reject $\frac{L}{2}$, s will become 0.
- $x = \frac{L}{6}$

- $x = \frac{20}{6} = \frac{10}{3}$
- $V = (2x - L)^2(x)$
- $V = \left(\frac{10}{3} - 20\right)^2 \left(\frac{10}{3}\right)$
- $V = 592.59 \text{ in}^2$

Cardboard to Largest Box Shortcut



- $x = \frac{L}{6}$
- $V = s^2 x = x(L - 2x)^2$

Example

Find the height of a right circular cylinder of maximum volume which can be inscribed in a sphere of radius 10 cm.

- $D = 2(10) = 20 \text{ cm}$
- $d = \sqrt{\frac{2}{3}} D$
- $h = \frac{1}{\sqrt{3}} D$
- $h = \frac{20}{\sqrt{3}} \text{ cm} = 11.55 \text{ cm}$

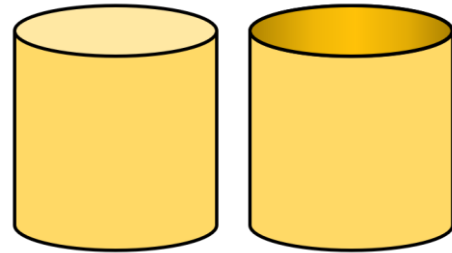
Example

Find the radius of a cylindrical measure and given a volume V so that the area of its sides and bottom shall be a minimum.

- $S_L \rightarrow \text{minimize}$
- $S_T = 2\pi r h + \pi r^2$
 - Only one circular base. (and bottom daw)
- $V = \pi r^2 h$

- $h = \frac{V}{\pi r^2}$
- $S_T = 2\pi r \left(\frac{V}{\pi r^2}\right) + \pi r^2$
- $S_T = \frac{2V}{r} + \pi r^2$
- $S'_T = -\frac{2V}{r^2} + 2\pi r = 0$
- $\frac{2V}{r^2} = 2\pi r$
- $\frac{r^2}{2V} = \frac{1}{2\pi r}$
- $r^3 = \frac{V}{\pi}$
- $r = \sqrt[3]{\frac{V}{\pi}}$
- $h = \frac{V}{\pi r^2} = \sqrt[3]{\frac{V}{\pi}} = r$

Least Amount of Material for Cylindrical Measure



- Open tank, just one base at bottom
 - $r = h = \sqrt[3]{\frac{V}{\pi}}$
- Closed tank, two base, with bottom and top lid
 - $d = h = \sqrt[3]{\frac{4V}{\pi}}$

Example

Postal regulations require that a parcel post package be no greater than 3 m in the sum of its length and girth (perimeter of the cross section). What is the volume in cu. m. of the largest package allowed by postal regulations if the package is to be rectangular in shape and has square ends?

- $4s + L = T$
 - $s = \text{side}$
 - $L = \text{length}$
 - $T = \text{Sum or Total}$
- $V \rightarrow \text{maximize}$
- $V = s^2 L$
- $L = T - 4s$
- $V = s^2(T - 4s)$

- $V = Ts^2 - 4s^3$
- $V' = 2Ts - 12s^2 = 0$
- $-s(12s - 2T) = 0$
- Reject $s = 0$
- $s = \frac{T}{6}$
- $s = \frac{3}{6} = \frac{1}{2}$
- $V = Ts^2 - 4s^3$
- $V = 0.25 \text{ m}^3$

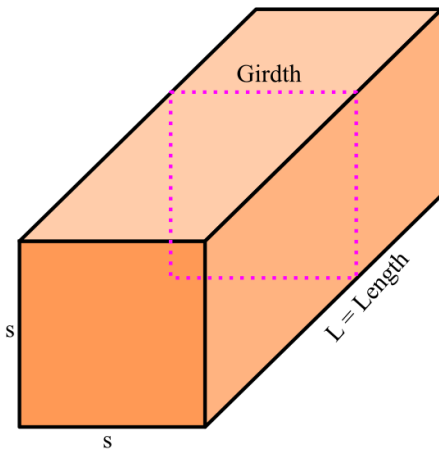
- $2x = 3y$
- $x = 1.5y$
- $x = 1.5(4)$
- $x = 6 \text{ kph}$

Example

For a certain specified sum, a man takes the contract to build a rectangular water tank, lined with lead. It has a square base and open top and holds 108 cu. m. What should be the height of the tank that requires the least quantity of lead?

- $s =$ side of square
- $h =$ height of tank
- $V = 108 \text{ cu. m.}$
- $S_T \rightarrow \text{minimize}$
- $V = s^2 h = 108$
- $h = \frac{V}{s^2}$
- $S_T = s^2 + 4sh$
- $S_T = s^2 + 4s\left(\frac{V}{s^2}\right)$
- $S_T = s^2 + \frac{4V}{s}$
- $S'_T = 2s - \frac{4V}{s^2} = 0$
- $\frac{4V}{s^2} = 2s$
- $4V = 2s^3$
- $s^3 = 2V$
- $s = \sqrt[3]{2V}$
- $h = \frac{V}{s^2} = \frac{V}{(2V)^{\frac{2}{3}}} = \frac{V^{\frac{1}{3}}}{2^{\frac{2}{3}}}$
- $h = \sqrt[3]{\frac{V}{4}}$
- $h = 3 \text{ m}$

Largest Volume Allowed by Postal Reg. Shortcut



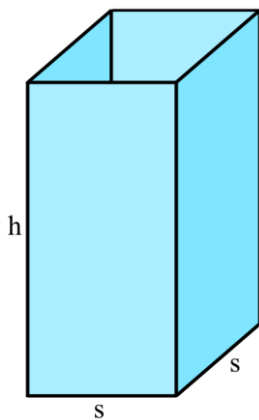
- $s = \frac{T}{6}$
- $L = \frac{T}{3}$
- $V = \frac{T^3}{108}$

Example

Assuming that the energy per hour expended in driving a steam boat through the water varies as the cube of the velocity, find the most economical rate per hour when steaming against a current running 4 kph.

- $E = kx^3$
- $v = x - y = x - 4$
- For economical rate per how we maximize the velocity per energy or the ratio velocity per energy.
- $R = \frac{v}{E} = \frac{x-y}{kx^3}$
- $R' = \frac{\frac{1}{k} \frac{x^3 - 3x^2(x-y)}{x^3}}{x^3} = 0$
- $x^3 - 3x^3 + 3x^2y = 0$
- $-2x^3 + 3x^2y = 0$
- $-x^2(2x - 3y) = 0$
- $2x - 3y = 0$

Minimize Material For Rectangular Tank Shortcut



- $h = \sqrt[3]{\frac{V}{4}}$
- $s = \sqrt[3]{\frac{V}{h}} = \sqrt[3]{2V}$
- $V = s^2 h$

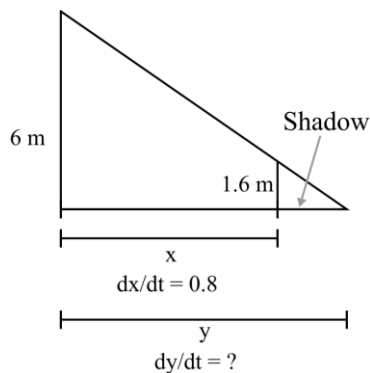
F) Related Rates Problems (Time Related Problems)

Steps for Solution

- Identify the quantity whose rate is required.
 - Q
- Relate the other quantities with known or given rates by the use of theorems or formulas.
 - $Q = f(x)$
- Evaluate derivative with respect to time.
 - $\frac{dQ}{dt} = f'(x) \frac{dx}{dt}$
- Solve for $\frac{dQ}{dt}$ by substituting given.

Example

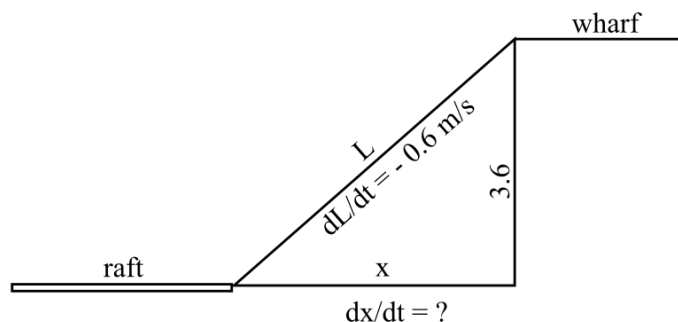
A boy 1.6 m tall is walking away from a street lamp 6 m tall at the rate of 0.8 m/s. Find how fast is the shadow moving.



- $\frac{y-x}{y} = \frac{1.6}{6}$
- $6y - 6x = 1.6y$
- $4.4y = 6x$
- $y = \frac{15}{11}x$
- $\frac{dy}{dt} = \frac{15}{11} \frac{dx}{dt}$
- $\frac{dy}{dt} = \frac{15}{11} (0.8)$
- $\frac{dy}{dt} = 1.09 \text{ m/s}$

Example

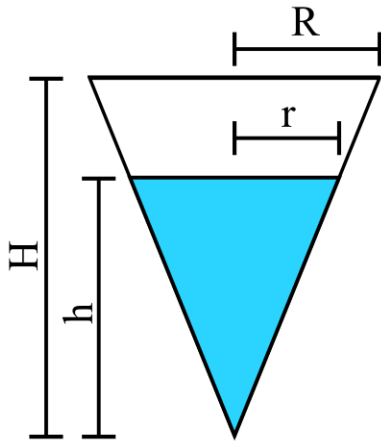
A man on a wharf 3.6 m above the level of water is pulling a rope tied to raft at 0.6 m/s. How fast is the raft approaching the wharf when there are 6 m of rope out?



- $x^2 + 3.6^2 = L^2$
- $2x \frac{dx}{dt} = 2L \frac{dL}{dt}$
- $2\sqrt{L^2 - 3.6^2} \frac{dx}{dt} = 2L \frac{dL}{dt}$
- Subs. $L = 6, \frac{dL}{dt} = -0.6$
- $\frac{dx}{dt} = -0.75 \frac{m}{s}$
- **The raft approaches wharf $0.75 \frac{m}{s}$.**

Example

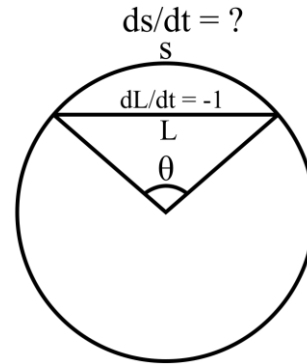
Water is running out of a conical funnel at the rate 1 cu. in/s. If the radius of the base of funnel is 4 in. and the altitude is 8 in., find the rate at which the water level is dropping when it is 2 in. from the top.



- $R = 4 \text{ in.}$
- $h = 8 \text{ in.}$
- $\frac{dV}{dt} = -1 \frac{\text{in}^3}{s}$
- $\frac{dh}{dt} = ?$
- $\frac{r}{R} = \frac{h}{H}$
- $\frac{r}{h} = \frac{R}{H} = \frac{4}{8}$
- $r = \frac{1}{2}h$ or $h = 2r$
- $V = \frac{1}{3}\pi r^2 h$
- $V = \frac{1}{3}\pi \left(\frac{1}{2}h\right)^2 h$
- $V = \frac{1}{12}\pi h^3$
- $\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$
- We do not know what h to use yet.
- $H - h = 2$
- $h = H - 2 = 8 - 2 = 6$
- Now we know $h = 6$ and $dV/dt = -1$. Substituting...
- $\frac{dh}{dt} = -\frac{1}{9\pi} \frac{\text{in}}{s} = -0.0354 \frac{\text{in}}{s}$

Example

The diameter of a circle is 10 cm. If the length of the chord is decreasing at the rate of 1 cm/s, find how fast is the arc subtended by the chord changing when the chord is 8 cm.



- $s = r\theta$
- $r = \frac{10}{2} = 5 \text{ cm}$
- $s = 5\theta$
- $\frac{ds}{dt} = 5 \frac{d\theta}{dt}$
- $\frac{L}{2} = r \sin \frac{\theta}{2}$
- $\frac{1}{2} \frac{dL}{dt} = \frac{5}{2} \cos \frac{\theta}{2} \frac{d\theta}{dt}$
- $L = 8 \text{ cm}$
- $\theta = 2 \arcsin \left(\frac{L}{2R} \right)$
- $\theta = 2 \arcsin \left(\frac{4}{5} \right)$
- Substituting $dL/dt = -1$ and $\theta = 2 \arcsin \left(\frac{4}{5} \right)$
- $\frac{d\theta}{dt} = -\frac{1}{3}$
- $\frac{ds}{dt} = 5 \left(-\frac{1}{3} \right)$
- $\frac{ds}{dt} = -\frac{5 \text{ cm}}{3 s} = -1.67 \frac{\text{cm}}{s}$

Example

The 2 sides of a triangle are 5 cm and 10 cm respectively and the acute angle between them is increasing at the rate of 5 degrees per minute. Find how fast is the 3rd side growing when the angle between the given 2 sides is 60°.

- $a = 5$ (constant)
- $b = 10$ (constant)
- $\frac{dc}{dt} = 5 \frac{\text{degree}}{s}$
- $\frac{dc}{dt} = ? @ C = 60^\circ$
- $c^2 = a^2 + b^2 - 2ab \cos C$
- Substituting...

- $c = \sqrt{75}$
- $2c \frac{dc}{dt} = -2ab (-\sin C) \frac{dC}{dt}$
- a
- $\frac{dc}{dt} = \frac{ab \sin C \frac{dC}{dt}}{c}$
- $\frac{dc}{dt} = \frac{(5)(10) \sin\left(\frac{\pi}{3}\right)\left(\frac{5\pi}{180}\right)}{\sqrt{75}}$
 - When an angle is used in an equation outside of a trigonometric function, always use rad format.
- $\frac{dc}{dt} = \frac{5}{36} \pi \frac{cm}{min} = 0.44 \frac{cm}{min}$
- **Shortcut:**
- $c^2 = a^2 + b^2 - 2ab \cos C$
- $c^2 = 125 - 100 \cos C$
- $c = \sqrt{125 - 100 \cos C}$
- Evaluate following derivative in calculator.
- $\frac{dc}{dt} = \frac{d}{dt} \sqrt{125 - 100 \cos C} \Big|_{C=\frac{\pi}{3}} \cdot \frac{dC}{dt}$
 - $\frac{dC}{dt} = \frac{\pi \text{ rad}}{36 \text{ s}}$
- $\frac{dc}{dt} = 0.44 \frac{cm}{s}$

G) The Differential

Differential

- Let $y = f(x)$
 - The differential is $dy = f'(x)dx$
 - differential = (derivative) dx
- Differential
 - Can be used to approximate change
 - $dx = f'(x)dx \approx f'(x)\Delta x = \Delta y$

Example

Find the approximate increase in volume of sphere if its radius increases from 2 to 2.05 cm.

- $V = \frac{4}{3}\pi r^3$
- $dV = 4\pi r^2 dr$
- $dV = 4\pi(2)^2(2.05 - 2)$
- $dV = 2.51 \text{ cm}^3$

Example

The volume of a cube is 8 m^3 . If the error in measuring its volume is 0.03 m^3 , find the error in measuring the edge of the given cube.

- $V = s^3$
- $8 = s^3 \rightarrow s = 2$
- $dV = 3s^2 ds$
- $0.03 = 3(2)^2 ds$
- $ds = 2.5 \text{ mm}$

Example

The specific gravity of body can be evaluated by the use of formula:

$$S = \frac{A}{A - W}$$

Where:

A = weight in air

W = weight in water

Find the maximum error in s if $A = 16N$, $W = 8N$ and $dA = dW = 0.01N$.

- $dS = ?$
- $dS = \frac{\partial S}{\partial A} dA + \frac{\partial S}{\partial W} dW$
- $\frac{\partial S}{\partial A} = \frac{(A-W)(1) - A(1)}{(A-W)^2} = -\frac{W}{(A-W)^2}$

- $\frac{\partial S}{\partial W} = \frac{(A-W)(0)-A(-1)}{(A-W)^2} = \frac{A}{(A-W)^2}$
- $ds = -\frac{W}{(A-W)^2} dA + \frac{A}{(A-W)^2} dW$
- Substitute. Negative sign is omitted since all terms must be positive for maximum error in s (maximum ds).
- $ds = 3.75 \times 10^{-3}$

Relative Error and Percent Error

- $RE = \frac{dQ}{Q}$
- $PE = \frac{dQ}{Q} \times 100$
- Where:
 - Q = Quantity
 - dQ = Approximate Error
- RE has no units.
 - Error mentioned in given without units is relative error.

Example

The error in measuring the radius of a sphere is 0.01. Find:

- Relative error in surface area, and
- Percent error in volume

- $RE_r = \frac{dr}{r} = 0.01$
- $S = 4\pi r^2$
- $dS = 4\pi(2r dr)$
- $RE_S = \frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r}$
- (a) $RE_S = 0.02$
- $V = \frac{4}{3}\pi r^3$
- $dV = 4\pi r^2 dr$
- $PE_V = \frac{dV}{V} \times 100 = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} \times 100$
- $PE_V = \frac{3dr}{r} \times 100$
- (b) $PE_V = 3\%$

H) Evaluating Limits

Evaluation of Limits

- Direct Substitution
- L'Hospital's Rule
 - $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = \dots$
 - If direct substitution results to indeterminate forms, derive the numerator and denominator.
 - Indeterminate forms:
 - $\frac{0}{0}$ or $\frac{\infty}{\infty}$
 - $0 \cdot \infty$
 - 0^0 or ∞^0 or 1^∞
 - $\frac{0}{\infty} = 0$; $\frac{\infty}{0} = \infty$
- By calculator:
 - Put the function using variable x in calculator and CALC (calculate) the expression at a value close to where the limit is evaluated.
 - $x \rightarrow 0: x = 0.0001$
 - $x \rightarrow 1: x = 0.9999$
 - $x \rightarrow \pi: x = 3.14$
 - $x \rightarrow e: x = 2.718$
 - $x \rightarrow \infty: x = 9999$

Example

Evaluate:

- $\lim_{x \rightarrow 4} \frac{x-4}{x^2-x-12}$
- $\frac{4-4}{4^2-4-12} = \frac{0}{0} \rightarrow \text{indeterminate}$
- $\lim_{x \rightarrow 4} \frac{1}{2x-1} = \frac{1}{2 \cdot 4 - 1} = \frac{1}{7}$

Example

Evaluate:

- $\lim_{x \rightarrow \infty} x \sin \frac{\pi}{x}$
- Calculate at $x = 9999\dots$
- $\lim_{x \rightarrow \infty} x \sin \frac{\pi}{x} \approx 3.1416$
- $\lim_{x \rightarrow \infty} x \sin \frac{\pi}{x} = \pi$

Example

Evaluate:

$$\lim_{x \rightarrow 1} (2 - x)^{\tan \frac{\pi x}{2}}$$

- Calculate at 0.9999
- $\lim_{x \rightarrow 1} (2 - x)^{\tan \frac{\pi x}{2}} \approx 1.89 = e^{2\pi}$
- **(Alternative method)**

Example

Evaluate:

$$\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\cos x}$$

- Negative at limit means approaching from left side, so the value should be near but at the left side.
- Evaluate at 89.99° .
- $\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\cos x} \approx 1.002$
- **$\lim_{x \rightarrow \frac{\pi}{2}^-} (\tan x)^{\cos x} = 1$**

Example

Evaluate:

$$\lim_{z \rightarrow e^{\frac{\pi}{4}i}} \frac{z^2}{z^4 + z + 1}$$

- $e^{\frac{\pi}{4}i} = 1 \angle \frac{\pi}{4}$
- Evaluate at $1 \angle \frac{\pi}{4}$ in complex mode calculator.
 - z^4 might need to be inputted as $(z^2)^2$
- **$\lim_{z \rightarrow e^{\frac{\pi}{4}i}} \frac{z^2}{z^4 + z + 1} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$**

Example

Evaluate:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x+y)}{3x+3y}$$

- Evaluate at (0.001,0.001) in calculator.
- $\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x+y)}{3x+3y} \approx 0.3333$
- **$\lim_{(x,y) \rightarrow (0,0)} \frac{\sin(x+y)}{3x+3y} = \frac{1}{3}$**

INTEGRAL CALCULUS

integral calculus

I) The Indefinite Integral

Indefinite Integral

- Integral without limits
- When evaluating, the answer contains a constant of integration C .
- Model:
 - $\int f(x)dx = F(x) + C$
 - $\int$ is the integral operator.
 - $f(x)$ is the integrand
 - $F(x)$ is the integral of the function $f(x)$
 - C is the constant of integration
- Can be:
 - Explicit
 - $y = f(x)$
 - Dependent variable (y) isolated
 - Implicit
 - $f(x, y) = 0$
 - Dependent variable (y) is not isolated

Integral of Differential

- Integral of differential of a function is the function itself.
- $\int d(f(x)) = f(x)$
- $\int d(x^2 + y^2) = x^2 + y^2$

Integral as an anti-derivative

- Power Rule:
 - Derivative: $\frac{d}{dx}(x^n) = nx^{n-1}$
 - Differential: $d(x^n) = nx^{n-1}dx$
 - Integrate both sides:
 - $\int nx^{n-1}dx = \int d(x^n)$
 - $\int nx^{n-1}dx = x^n$
 - $\int x^{n-1}dx = \frac{x^n}{n}$
 - $n \rightarrow n + 1$

○ Integral: $\int x^n dx = \frac{x^{n+1}}{n+1}$

- Product Rule
 - Derivative: $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$
 - Differential: $d(uv) = (u dv + v du)dx$
 - Integrate both sides.
 - $\int d(uv) = \int u dv + \int v du$
 - $uv = \int u dv + \int v du$
 - $\int u dv = uv - \int v du$
 - Integration by Parts Formula

Example

Evaluate the integral of the following:

$$\int 4e^{3 \ln x} dx$$

- $I = \int 4e^{3 \ln x} dx$
- $I = 4 \int e^{\ln x^3} dx$
- $I = 4 \int x^3 dx$
- $I = 4 \frac{x^{3+1}}{3+1} = x^4$
- $I = x^4 + C$

Example

Evaluate:

$$\int (7x^3 - 4x^2)dx$$

- $I = \int (7x^3 - 4x^2)dx$
- $I = \frac{7}{4}x^4 - \frac{4}{3}x^3 + C$

Example

Evaluate:

$$\int \frac{e^x dx}{\sqrt{25 - e^{2x}}}$$

- $u = e^x$
- $du = e^x dx$
- $u^2 = e^{2x}$
- $a^2 = 25$

- $I = \int \frac{du}{\sqrt{a^2 - u^2}}$
- $I = \arcsin\left(\frac{u}{a}\right) = +C$
- $I = \arcsin\left(\frac{e^x}{5}\right) + C$

Calculator Technique for Indefinite Integral

- $\int f(x)dx = F(x) + C$
- Derive both sides.
- $f(x) = \frac{d}{dx}(F(x))$
- Set a test value for x (say $x = a$)
- Evaluate the integrand at a using calculator.
 - $f(a)$
- Evaluate the integrals in the choices
 - $F(a)$

Calculator Technique for Previous Example

Evaluate:

$$\int \frac{e^x dx}{\sqrt{25 - e^{2x}}}$$

- $\arccos\left(\frac{e^x}{5}\right) + C$
- $\arccos(e^x) + C$
- $\arcsin(e^x) + C$
- $\arcsin\left(\frac{e^x}{5}\right) + C$

- $f(x) = \frac{e^x}{\sqrt{25 - e^{2x}}}$
- Evaluate at -2.
- $f(-2) = 0.027$
- Evaluating $\frac{d}{dx}(F(x))|_{x=3}$
 - For a: = -0.027
 - For b: = -0.14
 - For c: = 0.14
 - For d: = 0.027
- **Answer is d.**

II) The Definite Integral

Definite Integral

- Integral with limits
- When evaluating, answer is a constant.

Theorems for Definite Integrals

• Theorem 1

$$\int_a^b f(x)dx = [F(x)]_{x=a}^b = F(b) - F(a)$$

• Theorem 2

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

• Theorem 3

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

• Theorem 4

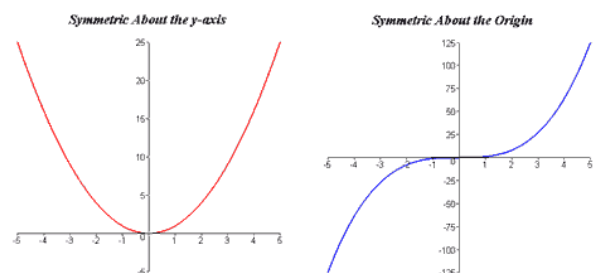
For even function $f(x)$

$$\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$$

For odd function $f(x)$

$$\int_{-a}^a f(x)dx = 0$$

Side Note:



• Even Function

- $f(-x) = f(x)$
- Symmetric about y-axis
- Even degree functions

• Odd Function

- $f(-x) = -f(x)$
- Symmetric about origin
- Odd degree functions

Walli's Formula

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{\{(m-1)(m-3) \dots 1 \text{ or } 2\} \{(n-1)(n-3) \dots 1 \text{ or } 2\}}{\{(m+n)(m+n-2) \dots 1 \text{ or } 2\}} \times \alpha$$

- $\alpha = \begin{cases} \frac{\pi}{2} & \text{both } m \text{ and } n \text{ are even} \\ 1 & \text{otherwise} \end{cases}$
 - If one is odd, then one.
 - If the two is even, then pi over two.
- Numerator
 - $m - 1$ tapos minus two ng minus two from there hanggang maging 1 or 2
 - Tapos times sa...
 - $n - 1$ tapos minus two ng minus two from there hanggang maging 1 or 2
- Denominator
 - $m + n$ tapos minus two ng minus two from there hanggang maging 1 or 2

The Forbidden Technique

- Definite Integrals can be solved using calculator... :O

Example

Evaluate the following:

$$\int_0^4 \frac{dx}{\sqrt{x}}$$

- $\int_0^4 \frac{dx}{\sqrt{x}} = \int_0^4 x^{-\frac{1}{2}} dx$
- $\int_0^4 \frac{dx}{\sqrt{x}} = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right]_{x=0}^4 = [2\sqrt{x}]_{x=0}^4$
- $\int_0^4 \frac{dx}{\sqrt{x}} = 2\sqrt{4} - 2\sqrt{0}$
- $\int_0^4 \frac{dx}{\sqrt{x}} = 4$
- (Alternatively...)
- Integral can be evaluated from 0.0001 to 4.
 - Why 0.0001? Because integrand is undefined at 0 which causes math error in calculator.

Example

Evaluate:

$$\int_0^{\pi/2} \sin^5 x \cos^3 x dx$$

- $I = \int_0^{\pi/2} \sin^5 x \cos^3 x dx$
- $m = 5$
- $n = 3$
- $\alpha = 1$
- $I = \frac{4 \cdot 2 \cdot 2}{8 \cdot 6 \cdot 4 \cdot 2}$
- $I = \frac{1}{24}$
- Alternatively, the integral can be put in the calculator.

Example

Evaluate:

$$\int_{-1}^1 (x + x^3 + x^5 + x^7 + \dots + x^{99}) dx$$

- Integrand $f(x)$ is 99th degree function which is an odd degree function. Thus, it is an odd function.
- By theorem...
- $\int_{-1}^1 f(x) dx = 0$

Example

Evaluate:

$$\int_2^{\infty} \frac{2dx}{x^3}$$

- Can be evaluated in calculator with upper limit being 999...
- $\int_2^{\infty} \frac{2dx}{x^3} = 0.25$

Example

Evaluate the line integral of $[\sqrt{y}dx + (x - y)dy]$ from (0,0) to (1,1)

- $(0,0) \rightarrow (x_1, y_1)$
- $(1,1) \rightarrow (x_2, y_2)$
- Line connecting two points can be derived using the two points form.
- $y - y_1 = \frac{x_2 - x_1}{y_2 - y_1} (x - x_1)$
- $y = x$

- From this, we can simplify the integrands to have only variables x for dx and only variables y for dy. The integrand/s becomes...
- $\sqrt{x}dx + 0dy$
- $\sqrt{x}dx$
- We use the limits x_1 to x_2
- $\int_0^1 \sqrt{x}dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_{x=0}^1$
- **Answer:** $\frac{2}{3}$

III) Multiple Integral

Multiple Integral

- Double Integral – Surface Integral
- Triple Integral – Volume Integral
- **Evaluate integral from inside out.**

Example

Evaluate the integral of $\frac{1}{x-y}dxdy$ with inner bounds 2y to 3y and outer bounds of 0 to 2.

- $\int_0^2 \int_{2y}^{3y} \frac{1}{x-y} dx dy$
- Integrating with respect to x (Integrating just the inner integral)
- $\int_{2y}^{3y} \frac{dx}{x-y} = [\ln(x-y)]_{x=2y}^{3y}$
- $\int_{2y}^{3y} \frac{dx}{x-y} = \ln 2y - \ln y = \ln\left(\frac{2y}{y}\right) = \ln 2$
- Integrating with respect to y (Putting back the outer integrals to the result of the inner integral)
- $\int_0^2 \ln 2 dy = [\ln 2 \cdot y]_{y=0}^2 = \ln 2 \cdot (2 - 0)$
- $\int_0^2 \ln 2 dy = 2 \ln 2 = \ln 2^2 = \ln 4$
- $\int_0^2 \int_{2y}^{3y} \frac{1}{x-y} dx dy = \ln 4$

Example

Evaluate:

$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta$$

- Evaluating inner integral...
- $\int_0^1 zr^2 \sin \theta dz = \left[\frac{1}{2}z^2 r^2 \sin \theta\right]_{z=0}^1$
- $\int_0^1 zr^2 \sin \theta dz = \frac{1}{2}r^2 \sin \theta (1^2 - 0^2)$

- $\int_0^1 zr^2 \sin \theta dz = \frac{1}{2}r^2 \sin \theta$
- Evaluating the next outer integral
- $\int_0^1 \int_0^1 zr^2 \sin \theta dz dr = \int_0^1 \left(\frac{1}{2}r^2 \sin \theta\right) dr$
- $\int_0^1 \int_0^1 zr^2 \sin \theta dz dr = \left[\frac{1}{6}r^3 \sin \theta\right]_{r=0}^1$
- $\int_0^1 \int_0^1 zr^2 \sin \theta dz dr = \frac{1}{6} \sin \theta (1^3 - 0^3)$
- $\int_0^1 \int_0^1 zr^2 \sin \theta dz dr = \frac{1}{6} \sin \theta$
- Evaluating outermost integral (whole integral)
- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta = \int_0^{\frac{\pi}{2}} \left(\frac{1}{6} \sin \theta\right) d\theta$
- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta = \left[-\frac{1}{6} \cos \theta\right]_{\theta=0}^{\frac{\pi}{2}}$
- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta = -\frac{1}{6} \left(\cos \frac{\pi}{2} - \cos 0\right)$
- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta = -\frac{1}{6} (0 - 1)$
- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta = \frac{1}{6}$

Calculator Technique for Multiple Integral

- If operation in integrand has only multiplication and division and no addition and subtraction, variables can be separated.

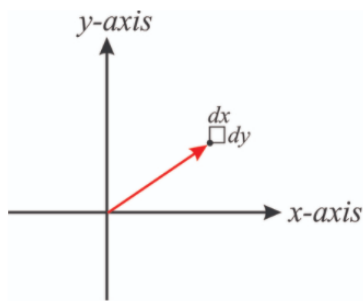
Example

(Same as previous, but we use the shorter method)

- $\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^1 zr^2 \sin \theta dz dr d\theta$
- $\int_0^{\pi/2} \sin \theta \int_0^1 r^2 dr \int_0^1 z dz$
- We can then input this in the calculator replacing all variables with x.
- We get...
- $\int_0^{\pi/2} \sin \theta \int_0^1 r^2 dr \int_0^1 z dz = \frac{1}{6}$

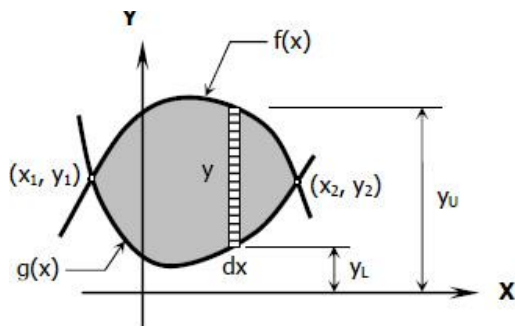
IV) Area in Rectangular Coordinates System

Differential Area



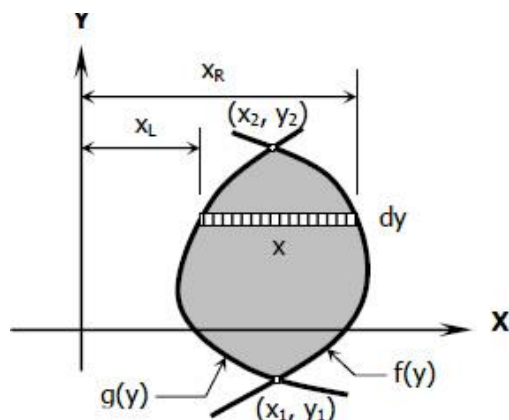
- $dA = dydx$
- $dA = xdy$

Vertical Element / Strip



- $A = \int_a^b (y_U - y_L) dx$

Horizontal Element / Strip



- $A = \int_a^b (x_R - x_L) dy$

Example

Find the area bounded by the curves $y = x^3$, $y = 8$, and $x = 0$.

- Using imagination
 - $y = x^3$

- Curves up from origin in QI.
- Curves down from origin in QIII.

- $y = 8$

- Horizontal line at $y = 8$
- Will intersect $y = x^3$ at $y = 8$ which is at $x = 2$

- $x = 0$

- y-axis
- Intersects $y = 8$ and $y = x^3$

- $y = 8$ is upper y.
- $y = x^3$ is lower y.
- Variable of integration is x. (dx)
- Limit is from 0 to 2
- Vertical Element
- $A = \int_0^2 (y_U - y_L) dx$
- $A = \int_0^2 (8 - x^3) dx$
 - Can already be evaluated in calculator.
- $A = \left[8x - \frac{x^4}{4} \right]_{x=0}^2$
- $A = 8(2 - 0) - \frac{1}{4}(2^4 - 0^4)$
- $A = 16 - 4 = 12$
- **$A = 12$ sq. units**

Example

Find the area bounded by the parabolas:

$$y = 6x - x^2$$

$$y = x^2 - 2x$$

- Using imagination:
 - The first parabola concaves down.
 - The second parabola concaves up.
 - Thus, the first parabola is the upper y and the second parabola is the lower y.
 - We get the intersection to know the limit of integration.
- Subtract the equations.
- $0 = -2x^2 + 8x$
- $2x^2 - 8x = 0$
- $2x(x - 4) = 0$
- $x = 0$
- $x = 4$
- $x = 0$

- $x = 4$
- Since we have upper and lower y's, we use a vertical strip.
- Variable of integration is x. (dx)
- Limits is from 0 to 4.
- $A = \int_0^4 (y_U - y_L) dx$
- $A = \int_0^4 (6x - x^2 - (x^2 - 2x)) dx$
- $A = \int_0^4 (-2x^2 + 8x) dx$
 - Can already be evaluated in calculator.
- $A = \left[-\frac{2}{3}x^3 + 4x^2 \right]_{x=0}^4$
- $A = -\frac{2}{3}(4^3 - 0^3) + 4(4^2 - 0^2)$
- $A = -\frac{2}{3}(64) + 64$
- $A = \frac{1}{3}(64)$
- $A = \frac{64}{3}$

Example

Find the area enclosed by the loop of $y^2 = 4x^2(1-x)$.

- Taking square root of both sides.
- $y = \pm 2x\sqrt{1-x}$
- One of these is upper and one is lower. The equation is hard to imagine, so let us just take the absolute value if we arrive to a negative value later. Assume + for upper and - for lower.
- $y_U = 2x\sqrt{1-x}$
- $y_L = -2x\sqrt{1-x}$
- Solving intersection points
- $y_U = y_L$
- $2x\sqrt{1-x} = -2x\sqrt{1-x}$
- $2x\sqrt{1-x} + 2x\sqrt{1-x} = 0$
- $4x\sqrt{1-x} = 0$
- $x = 0$
- $1-x = 0 \rightarrow x = 1$
- The limits are from 0 to 1.
- Using vertical element...
- $A = \int_0^1 (y_U - y_L) dx$
- $A = \int_0^1 (2x\sqrt{1-x} - (-2x\sqrt{1-x})) dx$
- Evaluating in Calculator: **$A = 1.067$**

Example

Find the area of the region bounded by $y = x^3$, $x = -2$, $x = 1$, and the x-axis.

- Using Imagination:
 - We have a curved line $y = x^3$ through the origin going from QIII to Q1.
 - There are two vertical lines $x = -2$ and $x = 1$
 - And the x-axis is $y = 0$
 - There arises two areas. One in QIII and one in QI.
 - In QIII, A_1
 - The upper y is $y = 0$.
 - The lower y is $y = x^3$.
 - In QI, A_2
 - The upper y is $y = x^3$.
 - The lower y is $y = 0$.
- $A = A_1 + A_2$
- $A = \int_{-2}^0 (0 - x^3) dx + \int_0^1 (x^3 - 0) dx$
- $A = -\int_{-2}^0 x^3 dx + \int_0^1 x^3 dx$
- $A = -\left[\frac{x^4}{4}\right]_{x=-2}^0 + \left[\frac{x^4}{4}\right]_{x=0}^1$
- $A = -\frac{1}{4}(0^4 - (-2)^4) + \frac{1}{4}(1^4 - 0^4)$
- $A = 4 + \frac{1}{4}$
- $A = \frac{17}{4}$
- **$A = 4.25 \text{ sq. units}$**

Shortcut for Area Bounded by Curve and X-axis

- $A = \int_a^b |y| dx$

Example

- (Same as previous, but we use the shorter method)
- $y = x^3$
- $A = \int_{-2}^1 |x^3| dx$
- Evaluating in calculator...
- **$A = 4.25 \text{ sq. units}$**

V) Area Bounded by Parametric Equations

Example

Find the area bounded by the curve at the 2nd and 3rd quadrants.

$$x = t^2 - 1$$

$$y = 5t^3(t^2 - 1)$$

- $dx = 2t dt$
- $y = 5t^5 - 5t^3$
- $x = 0$
- $t^2 - 1 = 0$
- $t = \pm 1$
- $A = \int y dx$
- $A = \int_{-1}^1 (5t^5 - 5t^3) 2t dt$
- Evaluate in calculator.
- $A = -\frac{8}{7}$
- Take absolute value.
- $A = \frac{8}{7}$ sq. units

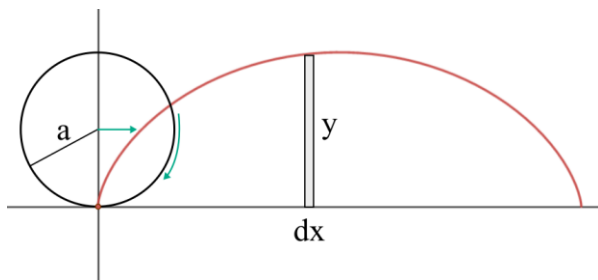
Example

Find the area bounded by one arc of cycloid:

$$x = a(\theta - \sin \theta), y = a(1 - \cos \theta)$$

And x-axis.

Note: Cycloid – locus of points made by a fixed point in a circle as it rolls on a straight line (such as the x-axis in this case).

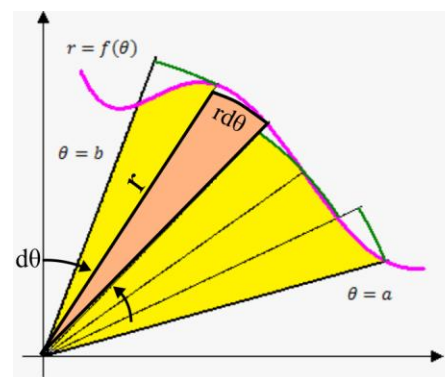


- The point of the circle will touch the x-axis again when it goes one full revolution.
- That happens when it already travelled on the x-axis the length of its circumference.
- Thus, the point where it intersects the x-axis again is $(2\pi a, 0)$
- $dx = a(1 - \cos \theta) d\theta$
- $A = \int y dx$
- $A = \int a(1 - \cos \theta) a(1 - \cos \theta) d\theta$

- $A = \int a^2 (1 - \cos \theta)^2 d\theta$
- Getting the limits...
 - $x = a(\theta - \sin \theta)$
 - $x: 0 \text{ to } 2\pi a \rightarrow \theta: 0 \text{ to } 2\pi$
- $A = \int_0^{2\pi} a^2 (1 - \cos \theta)^2 d\theta$
- $A = a^2 \int_0^{2\pi} (1 - \cos \theta)^2 d\theta$
- Evaluating integral in calculator.
- $A = 3\pi a^2$

VI) Area in Polar Coordinates System

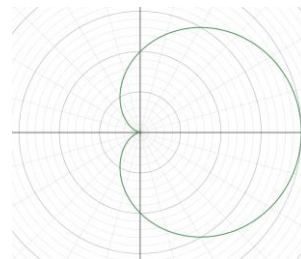
Differential Area



- $dA = \frac{1}{2} r^2 d\theta$
- $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Example

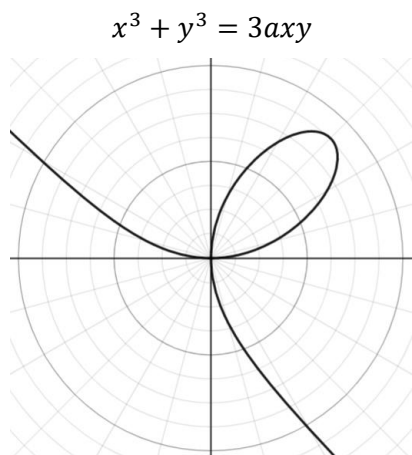
Find the area bounded by the cardioid $r = a(1 + \cos \theta)$



- $A = \frac{1}{2} \int_0^{2\pi} (a(1 + \cos \theta))^2 d\theta$
- $A = \frac{1}{2} a^2 \int_0^{2\pi} (1 + \cos \theta)^2 d\theta$
- Evaluate integral in calculator.
 - Integral results to 3π
- $A = \frac{3\pi a^2}{2}$

Example

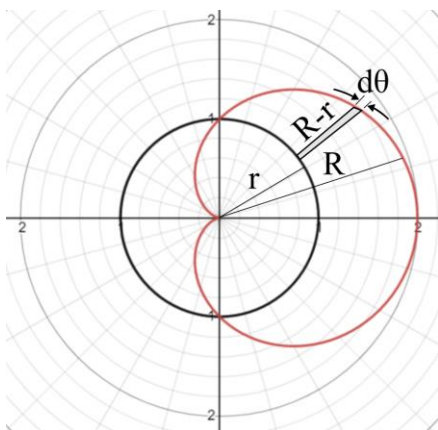
Find the area of the loop made by the folium of Descartes:



- Convert to polar coordinate system.
- $x = r \cos \theta$
- $y = r \sin \theta$
- $r^3(\cos^3 \theta + \sin^3 \theta) = 3ar^2 \cos \theta \sin \theta$
- $r = \frac{3a \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta}$
- $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$
- $A = \frac{1}{2} \int_0^{\pi/2} \left(\frac{3a \cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta} \right)^2 d\theta$
- $A = \frac{9}{2} a^2 \int_0^{\pi/2} \left(\frac{\cos \theta \sin \theta}{\cos^3 \theta + \sin^3 \theta} \right)^2 d\theta$
- Evaluating the integral using calculator.
 - Integral evaluates to $\frac{1}{3}$
- $A = \frac{9}{2} a^2 \cdot \frac{1}{3}$
- $A = \frac{3}{2} a^2$

Example

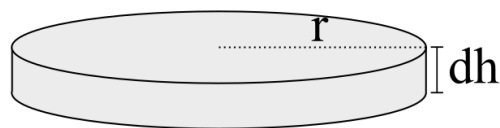
Find the area outside the circle $r = 1$ and inside the cardioid $r = 1 + \cos \theta$.



- R and r are like r_{out} and r_{in}
- $R = 1 + \cos \theta$
- $r = 1$
- To get limits, find intersection
- $R = r$
- $1 + \cos \theta = 1$
- $\cos \theta = 0$
- $\alpha = -\frac{\pi}{2}$
- $\beta = \frac{\pi}{2}$
- $A = \frac{1}{2} \int_{\alpha}^{\beta} (R^2 - r^2) d\theta$
- $A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} ((1 + \cos \theta)^2 - 1^2) d\theta$
- $A = \frac{1}{2} \int_{-\pi/2}^{\pi/2} ((1 + \cos \theta)^2 - 1) d\theta$
- $A = \int_0^{\pi/2} ((1 + \cos \theta)^2 - 1) d\theta$
- Evaluating in calculator...
- $A = 2.79$

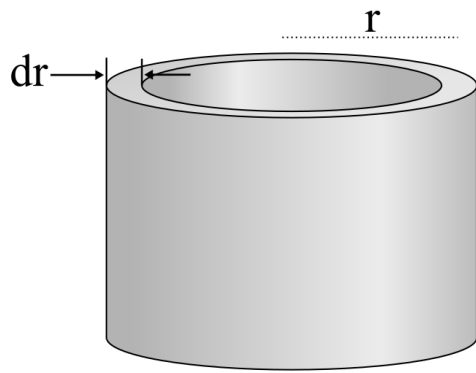
VII) Volume of Solids of Revolution

Disk Method

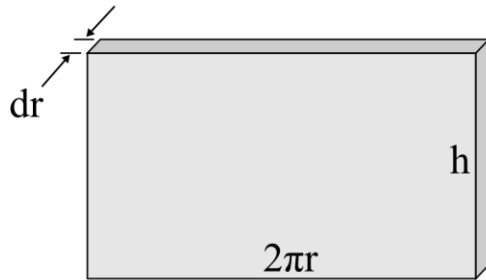


- $dV = \pi r^2 dh$
- Pag may butas...
- $dV = \pi(r_{out}^2 - r_{in}^2) dh$

Cylindrical Shell Method



(Cutting the cylindrical shell, and laying it flat)



- $dV = 2\pi r h dr$
- Pag may butas...
- $dV = 2\pi r(h_U - h_L)dr$

Example

Find the volume of the solid generated when the area bounded by the curve $y = x^3$, $y = 8$, and $x = 0$ is revolved about the y -axis.

- Determining limits of integration without sketch...
 - $y = x^3$
 - $y = 8$
 - $x^3 = 8$
 - $x = \pm 2$
 - Reject -2 because area is in QI.
 - $x = 2$
 - Limits of integration is $x = 0$ and $x = 2$
 - For integrating with respect to y , $y = 0$ and $y = 8$
- (By disk method)
- The radius is x
- Disk thickness / height is dy .
- $A = \int_0^8 \pi(x^2) dy$
- $y = x^3$
- $y^{2/3} = x^2$

- $A = \pi \int_0^8 y^{2/3} dy$
- $A = \pi \left[\frac{3}{5} y^{5/3} \right]_{y=0}^8$
- $A = \frac{3\pi}{5} \left(8^{5/3} - 0^{5/3} \right)$
- $A = \frac{3\pi}{5} (2^5)$
- $A = \frac{96}{5} \pi \text{ cu. units}$
- (By cylindrical method)
- The radius is still x .
- The height of cylinder is $y_U - y_L$
- Upper y is $y = 8$
- Lower y is $y = x^3$
- Thickness of cylinder is dx
- $A = \int_0^2 2\pi x(8 - x^3) dx$
- $A = 2\pi \int_0^2 (8x - x^4) dx$
- $A = 2\pi \left[4x^2 - \frac{1}{5}x^5 \right]_{x=0}^2$
- $A = 2\pi \left[4(2^2 - 0^2) - \frac{1}{5}(2^5 - 0^5) \right]$
- $A = 2\pi \left(16 - \frac{32}{5} \right)$
- $A = \frac{96}{5} \pi \text{ cu. units}$

Example

Find the volume generated when the area bounded by $y^2 = 12x$ and the line $x = 3$ is revolved about the line $x = 3$.

- $y^2 = 12(3) = 36$
- $y = \pm 6$
- Use disk method.
- Radius is $3 - \frac{y^2}{12}$
- $A = \int_{-6}^6 \pi(x_R^2 - x_L^2) dy$
- $A = \int_{-6}^6 \pi \left(3 - \frac{y^2}{12} \right)^2 dy$
- Evaluating using calculator...
- $A = \frac{288}{5} \pi$

Example

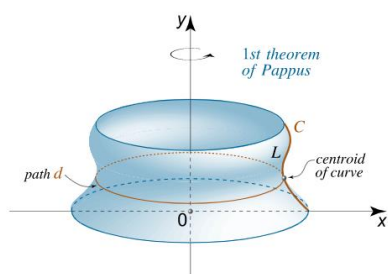
The area bounded by the parabola $x^2 = 4y$ and the line $x = 4 - 2y$ revolves about the line $x = 3$. Find the volume generated.

- $2y = 4 - x$
- $4y = 8 - 2x$
- $x^2 = 8 - 2x$
- $x^2 + 2x - 8 = 0$
- $x = 2, -4$
- Cylindrical method looks easier when imagined since there is a hole in the disk if disk method is used.
- Radius is $3 - x$.
- Height is Upper minus lower is $\frac{4-x}{2} - \frac{x^2}{4}$
- Thickness of cylinder is dx
- $A = \int_{-4}^2 2\pi(3-x) \left(\frac{4-x}{2} - \frac{x^2}{4} \right) dx$
- Evaluating in calculator...
- **$A = 72\pi$ cu. units**

VIII) Volume of Solids of Revolution by Second Theorem of Pappus

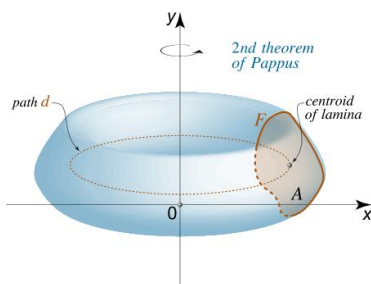
Second Theorem of Pappus

- First Theorem (Surface Area)



$$A = Ld = 2\pi\bar{r}L$$

- Second Theorem (Volume)

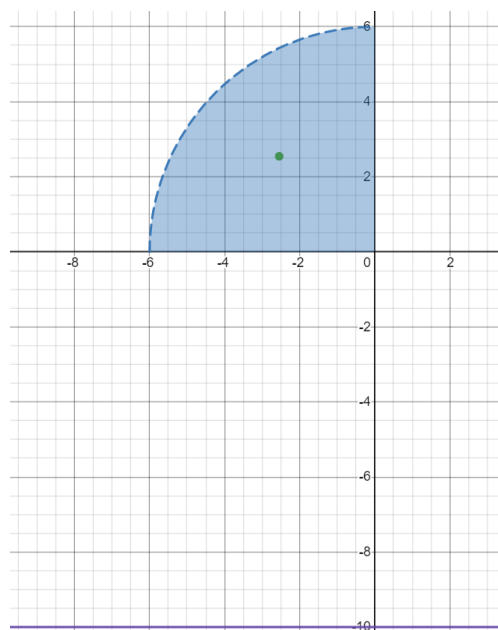


$$V = Ad = 2\pi\bar{r}A$$

- $\bar{r}$ = radius of circular path travelled by centroid

Example

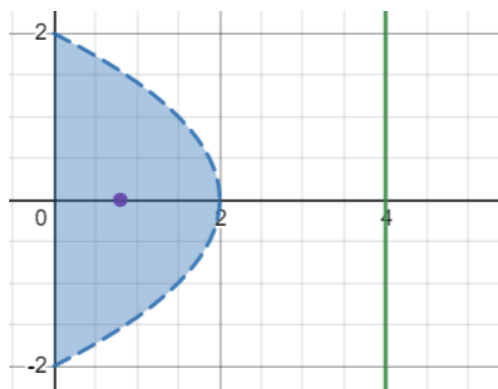
Find the volume of the solid generated when the area of the circle $x^2 + y^2 = 36$ on the 2nd quadrant is revolved about this line $y + 10 = 0$.



- Centroid of a quarter-circle is $\frac{4r}{3\pi}$ away from center, in both x and y-directions.
- $V = Ad$
- $V = \left(\frac{1}{4}\pi(6)^2\right) \left(2\pi \left(10 + \frac{4(6)}{3\pi}\right)\right)$
- **$V = 2228.92$ cu. units**

Example

The area bounded by the parabola $y^2 = -2(x - 2)$ and $x = 0$ is revolved about the line $x = 4$. Find the volume generated.



- Centroid of parabola is $\frac{2}{5}h$ from the base.
- $h = 2$
- Centroid is $\frac{4}{5}$ from the base.
- $\bar{r} = 4 - \frac{4}{5} = \frac{16}{5}$

- $V = Ad$
- $V = \left(\frac{2}{3}bh\right)(2\pi\bar{r})$
- $V = \left(\frac{2}{3}(4)(2)\right)\left(2\pi\left(\frac{16}{5}\right)\right)$
- $V = \frac{512}{15}\pi$ cu. units

Example

Find the volume generated by rotating a circle:

$$x^2 + y^2 + 6x + 4y + 12 = 0$$

About the y-axis.

- $(x+3)^2 + (y+2)^2 = -12 + 9 + 4$
- $(x+3)^2 + (y+2)^2 = 1^2$
- $r = 1$
- Center $C(-3, -2)$
- $\bar{r} = 0 - (-3) = 3$
 - x of y-axis minus x of center
- $V = Ad$
- $V = (\pi(1)^2)(2\pi(3))$
- $V = 6\pi^2 = 59.22$ cu. units

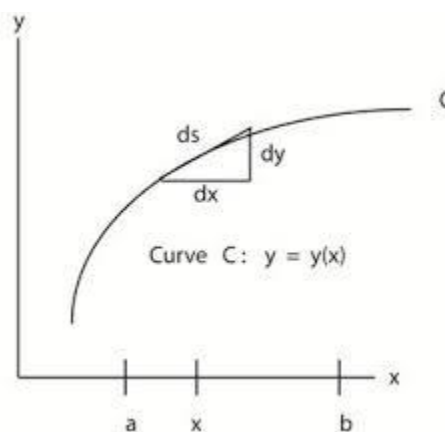
Example

The area enclosed by the ellipse $4x^2 + 9y^2 = 36$ is revolved about the line $x = 3$. What is the volume generated?

- $\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$
- $\bar{r} = 3$
 - Distance of center (0,0) and $x = 3$
- $V = Ad = (\pi ab)(2\pi\bar{r})$
- $V = (\pi \cdot 3 \cdot 2)(2\pi \cdot 3)$
- $V = 36\pi^2 = 355.31$ cu. units

IX) Arc Length

Arc Length



- $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$ (Rectangular C.S.)
- $s = \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$
 - $ds^2 = dx^2 + dy^2$
 - $ds = \sqrt{dx^2 + dy^2}$
 - $ds = \sqrt{dx^2 + dy^2} \cdot \frac{dx}{dx}$
 - $ds = \sqrt{\frac{dx^2 + dy^2}{dx^2}} dx$
 - $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
 - $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
- $s = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$ (Polar C.S.)
 - $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
 - $x = r \cos \theta$
 - $dx = -r \sin \theta d\theta + \cos \theta dr$
 - $y = r \sin \theta$
 - $dy = r \cos \theta d\theta + \sin \theta dr$
 - $ds = \sqrt{dx^2 + dy^2}$
 - $ds = \sqrt{(-r \sin \theta d\theta + \cos \theta dr)^2 + (r \cos \theta d\theta + \sin \theta dr)^2}$
 - $ds = \sqrt{r^2 d\theta^2 + dr^2}$
 - $ds = \sqrt{r^2 d\theta^2 + dr^2} \cdot \frac{d\theta}{d\theta}$

- $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$
- $s = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$
- $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$ (Parametric Eq'n)
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
- $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
- $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
- $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dx \cdot \frac{dt}{dx}$
- $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$
- $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

Example

Find the length of the arc described by $x^2 = 4y$ from $x = -2$ to $x = 2$.

- Since limits for x are given, use:

- $s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$
- $s = \int_{-2}^2 \sqrt{1 + \left(\frac{d}{dx}\left(\frac{x^2}{4}\right)\right)^2} dx$
- $s = \int_{-2}^2 \sqrt{1 + \left(\frac{1}{2}x\right)^2} dx$
- $s = \int_{-2}^2 \sqrt{\frac{x^2}{4} + 1} dx$
- Evaluate in calculator...
- **$s = 4.59$ units**

Example

Find the length of the curve described by the cardioid $r = 2(1 + \cos \theta)$.

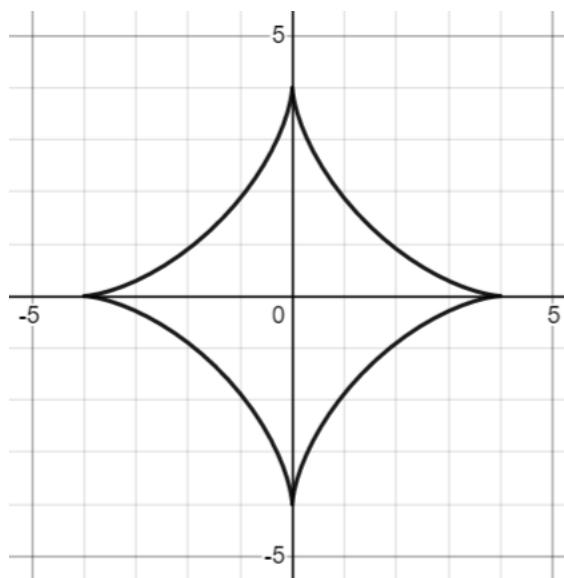
- $s = \int_a^b \sqrt{r^2 + (r')^2} d\theta$
- $s = \int_0^{2\pi} \sqrt{4(1 + \cos \theta)^2 + 4 \sin^2 \theta} d\theta$
 - Evaluating in calculator...X
- **$s = 16$ units**

Example

Find the length of the curve described by the 4-cusps hypocycloid or astroid:

$$x = 4 \cos^3 t$$

$$y = 4 \sin^3 t$$

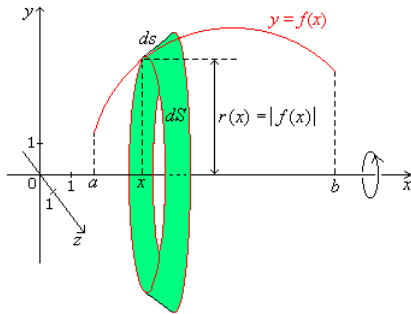


- Limits is 0 to 2π or just 0 to $\frac{\pi}{2}$ to get one quadrant, then just multiply by 4 to get the length of curve in all quadrants.
- $s = 4 \int_0^{\pi/2} \sqrt{(x')^2 + (y')^2} dx$
- $s = 4 \int_0^{\pi/2} \sqrt{(12 \cos^2 t \sin t)^2 + (12 \sin^2 t \cos t)^2} dx$
- Evaluating in calculator...
- **$s = 24$ units**

X) Surface Area of Solids of Revolution

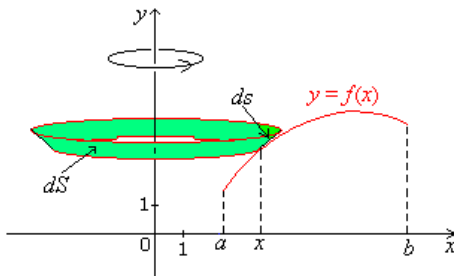
Surface Area Formulas

- Revolution about x-axis



- $S = 2\pi \int_{x=a}^{x=b} y ds$
- $ds = \sqrt{1 + (y')^2} dx$
- $S = 2\pi \int_a^b y \sqrt{1 + (y')^2} dx$

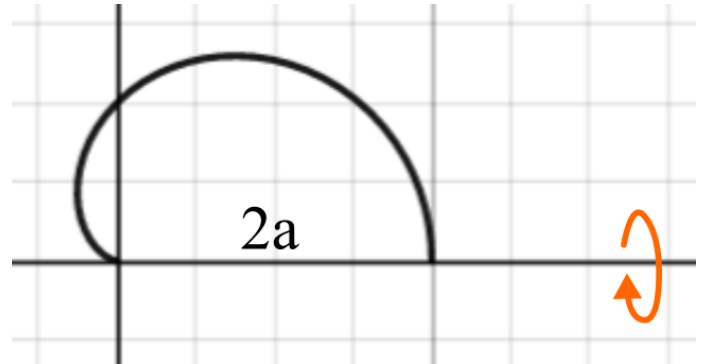
- Revolution about y-axis



- $S = 2\pi \int_{x=a}^{x=b} x ds$
- $ds = \sqrt{1 + (x')^2} dy$
- $S = 2\pi \int_{y(a)}^{y(b)} x \sqrt{1 + (x')^2} dy$

Example

Find the surface area formed by revolving the cardioid $r = a(1 + \cos \theta)$ about the x-axis.



- $S = \int 2\pi y ds$
- $S = 2\pi \int r \sin \theta \sqrt{r^2 + (r')^2} d\theta$
- $S = 2\pi \int_0^\pi a(1 + \cos \theta)(\sin \theta) \sqrt{a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta} d\theta$
- $S = 2\pi a^2 \int_0^\pi (1 + \cos \theta)(\sin \theta) \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} d\theta$
- Evaluating in Calculator....
- $S = 2\pi a^2 \cdot \frac{16}{5}$
- $S = \frac{32}{5} \pi a^2$

Example

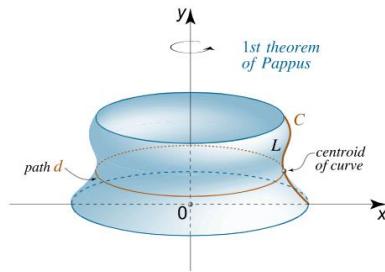
Find the surface area are generated when the parabola $y^2 = x$ from (0,0) to (1,1) is revolved about the x-axis.

- $S = \int 2\pi y ds$
- $S = 2\pi \int_0^1 y \sqrt{1 + (y')^2} dx$
- $S = 2\pi \int_0^1 \sqrt{x} \sqrt{1 + \left(\frac{1}{2\sqrt{x}}\right)^2} dx$
- $S = 2\pi \int_0^1 \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx$
- $S = \pi \int_0^1 \sqrt{4x + 1} dx$
- Evaluating in calculator...
- $S = 5.33$

XI) Surface Area of Solids of Revolution

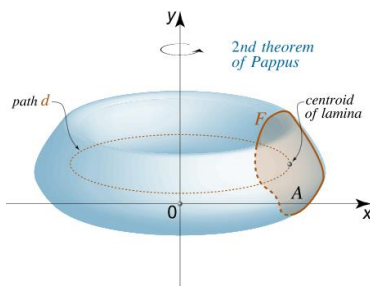
First Theorem of Pappus

- First Theorem (Surface Area)



$$\circ A = Ld = 2\pi\bar{r}L$$

- Second Theorem (Volume)



$$\circ V = Ad = 2\pi\bar{r}A$$

- $\bar{r}$ = radius of circular path travelled by centroid

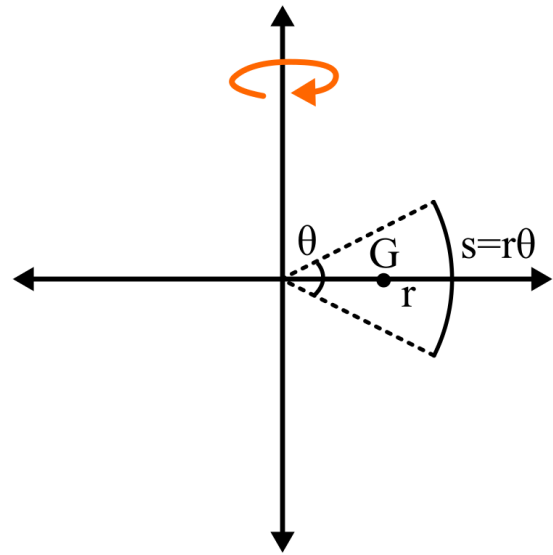
Example

Find the surface area generated by revolving a semi-circular arc on the 1st and 2nd quadrants $x^2 + y^2 = 16$ about the x-axis.

- $A = Ld$
- $A = (\pi r)(2\pi\bar{r})$
- $A = (\pi r)\left(2\pi \cdot \frac{2r}{\pi}\right)$
- $A = (\pi r)(4r)$
- $A = 4\pi r^2$ (Surface Area of sphere)
- $A = 4\pi(4)^2$
- $A = 64\pi$ sq. units

Example

Find the surface area generated by the circular arc symmetric with respect to x-axis, subtended by angle θ and with radius r when revolved about the y-axis.



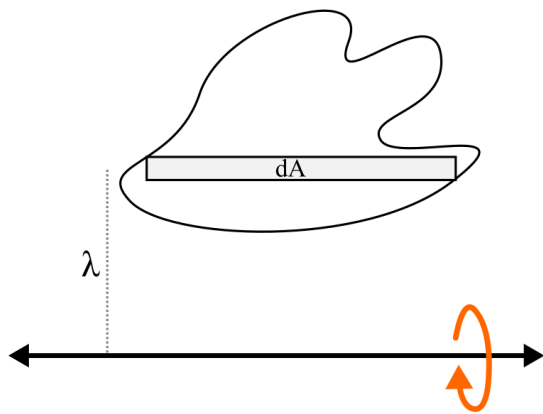
- $\bar{r} = \frac{2R}{\theta} \sin \frac{\theta}{2}$
- $A = Ld$
- $A = (r\theta)(2\pi\bar{r})$
- $A = (r\theta)\left(2\pi \cdot \frac{2r}{\theta} \sin \frac{\theta}{2}\right)$
- $A = 4\pi r^2 \sin \frac{\theta}{2}$
- (Alternative Method using formula for Zone)
- $A = 2\pi r h$
- $A = 2\pi r \left(2r \sin \frac{\theta}{2}\right)$
- $A = 4\pi r^2 \sin \frac{\theta}{2}$

XII) Moment of Inertia of Area

Moment of Inertia

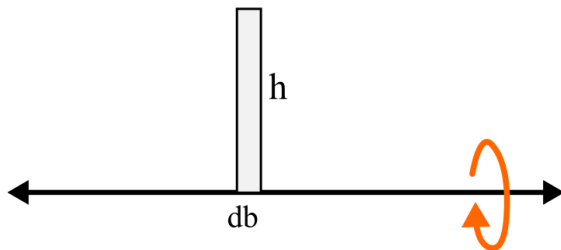
- Tendency to resist angular acceleration.
- Resistance of material to flexure and twist

Parallel Axis Method



- $I = \int_A \lambda^2 dA$
- Differential area is parallel to axis of moment.

Perpendicular Axis Method

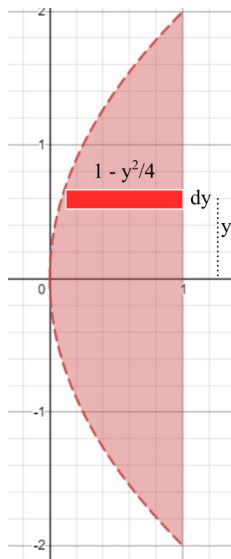


- $I = \frac{1}{3} \int_B h^3 db$
- Differential area is perpendicular to axis of moment.

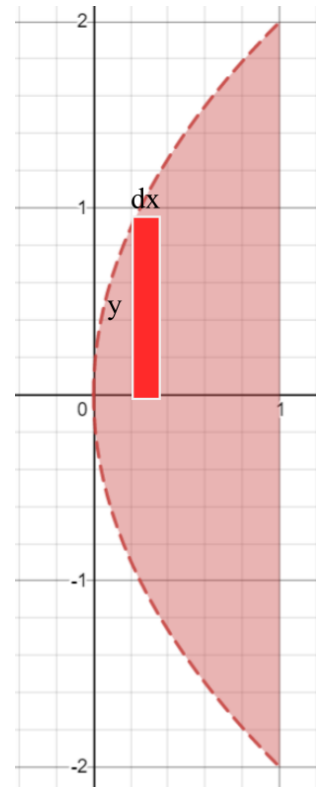
Example

Find the moment of inertia of the area bounded by $y^2 = 4x$ and the line $x = 1$ with respect to x-axis.

- **Parallel axis method**



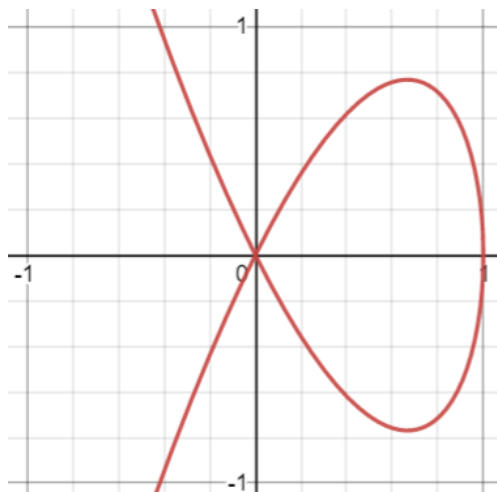
- $I_x = \int_{-2}^2 y^2 \left(1 - \frac{y^2}{4}\right) dy$
- $I_x = 2.1333 \dots = \frac{32}{15}$
- Perpendicular axis method:



- $I_x = \frac{1}{3} \int (y^3 - (-y^3)) dx$
- $I_x = \frac{2}{3} \int y^3 dx$
- $I_x = \frac{2}{3} \int_0^1 (4x)^{\frac{3}{2}} dx$
- $I_x = 2.1333 \dots = \frac{32}{15}$

Example

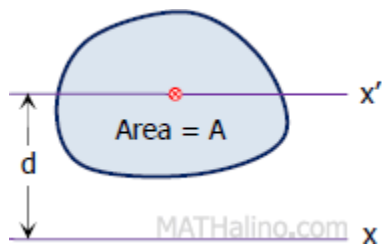
Find the moment of inertia of the area of the loop of the curve $y^2 = 4x^2(1-x)$ with respect to y-axis.



- $I_y = \int x^2 dA$
- $y_{U,L} = \pm 2x\sqrt{1-x}$
- $I_y = \int_0^1 x^2 (y_U - y_L) dx$
- $I_y = \int_0^1 x^2 (4x\sqrt{1-x}) dx$
- $I_y = 0.4063$

XIII) Transfer Formula for Moment of Inertia

Transfer Formula for Moment of Inertia



(x' is centroidal axis or neutral axis)

- $I_x = \bar{I}_x + Ad^2$
 - $\bar{I}_x$ = centroidal moment of inertia
 - I_x = moment of inertia about line x

Shape	Moment of Inertia	Radius of Gyration
Rectangle 	$\bar{I}_x = \frac{bh^3}{12}$ $I_x = \frac{bh^3}{3}$	$\bar{k}_x = \frac{h}{\sqrt{12}}$ $k_x = \frac{h}{\sqrt{3}}$
Triangle 	$\bar{I}_x = \frac{bh^3}{36}$ $I_x = \frac{bh^3}{12}$	$\bar{k}_x = \frac{h}{\sqrt{18}}$ $k_x = \frac{h}{\sqrt{6}}$
Circle 	$\bar{I}_x = \frac{\pi r^4}{4}$ $\bar{J} = \frac{\pi r^4}{2}$	$\bar{k}_x = \frac{r}{2}$ $\bar{k}_z = \frac{r}{\sqrt{2}}$
Semicircle 	$I_x = \bar{I}_y = \frac{\pi r^4}{8}$ $\bar{I}_x = 0.11r^4$	$k_x = \bar{k}_y = \frac{r}{2}$ $\bar{k}_x = 0.264r$
Quarter circle 	$I_x = I_y = \frac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = 0.055r^4$	$k_x = k_y = \frac{r}{2}$ $\bar{k}_x = \bar{k}_y = 0.264r$
Ellipse 	$\bar{I}_x = \frac{\pi ab^3}{4}$ $\bar{I}_y = \frac{\pi a^3 b}{4}$	$\bar{k}_x = \frac{b}{2}$ $\bar{k}_y = \frac{a}{2}$

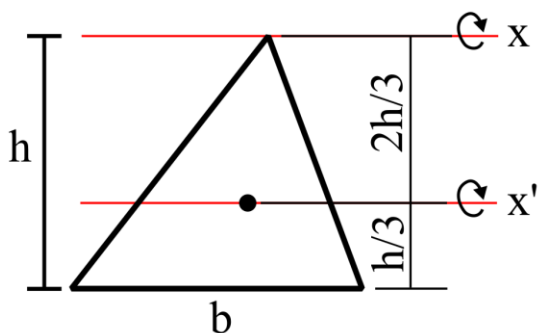
Example

Find the moment of inertia of the area enclosed by $x^2 + y^2 = 36$ with respect to the line $y = 8$.

- $r = 6$
- $d = 8$
- $\bar{I}_x = \frac{\pi r^4}{4}$
- $I_x = \bar{I}_x + Ad^2$
- $I_x = \frac{\pi}{4}(6)^4 + \pi(6)^2(8)^2$
- $I_x = 2628\pi = 8256.11 \text{ units}^4$

Example

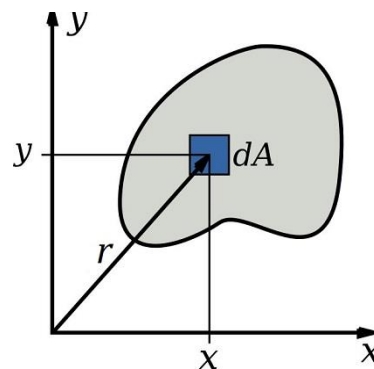
Find the moment of inertia of a triangle with base b and height h with respect to its vertex.



- $d = \frac{2}{3}h$
- $\bar{I}_x = \frac{bh^3}{12}$
- $I_x = \bar{I}_x + Ad^2$
- $I_x = \frac{bh^3}{12} + \left(\frac{1}{2}bh\right)\left(\frac{2}{3}h\right)^2$
- $I_x = \frac{bh^3}{36} + \frac{2bh^3}{9}$
- $I_x = \frac{bh^3}{4}$

XIV) Polar Moment of Inertia

Polar Moment of Inertia



- Tendency to resist torsion
- $J = \int_A r^2 dA = \frac{1}{4} \int_{\theta_1}^{\theta_2} r^4 d\theta$
- $J = I_x + I_y$

Example

Find the polar moment of inertia of circle with radius a and center at origin.

- $r = a$
- $J = \frac{1}{4} \int_{\theta_1}^{\theta_2} r^4 d\theta$
- $J = \frac{1}{4} \int_0^{2\pi} a^4 d\theta$
- $J = \frac{1}{4} [a^4 \theta]_{\theta=0}^{2\pi}$
- $J = \frac{1}{4} a^4 (2\pi - 0)$
- $J = \frac{\pi}{2} a^4$

Examples

Find the polar moment of inertia I_o or J of the area bounded by $r^2 = a^2 \cos \theta$.

- $r^4 = a^4 \cos^2 \theta$
- $J = \frac{1}{4} \int_{\theta_1}^{\theta_2} r^4 d\theta$
- $J = \frac{1}{4} \int_0^{2\pi} (a^4 \cos^2 \theta) d\theta$
- $J = \frac{1}{4} a^4 \int_0^{2\pi} (\cos^2 \theta) d\theta$
- $J = \frac{\pi}{4} a^4$

XV) Moment of Inertia for Volume

Parallel Axis Method

- $I = \int_V r^2 dV$
 - $dV = 2\pi r L dr$
 - Cylindrical Shell Method

Perpendicular Axis Method

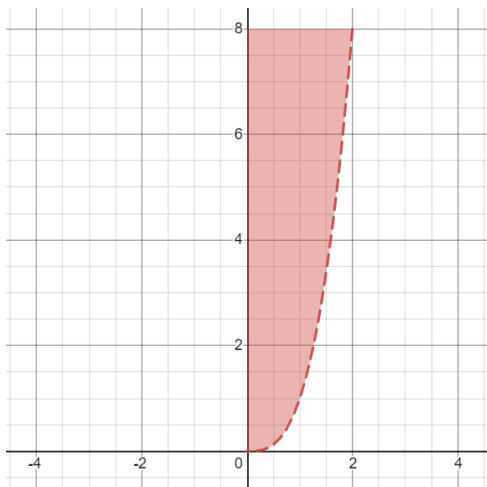
- $I = \frac{1}{2} \int_V r^2 dV$
 - $dV = \pi r^2 dh$
 - Disk Method

Example

Find the moment of inertia of the solid generated by revolving the area bounded by $y = x^3$, $y = 8$ and y -axis:

- (a) With respect to x -axis
- (b) With respect to y -axis

Note: Axis of Moment is axis of revolution.



- (a) About x -axis
 - Use horizontal element and cylindrical shell method.
 - $dV = 2\pi y x dy$
 - $dV = 2\pi y y^{\frac{1}{3}} dy$
 - $I_x = \int_V r^2 dV$
 - $I_x = \int_0^8 y^2 2\pi y y^{\frac{1}{3}} dy$
 - $I_x = 2\pi \int_0^8 y^3 y^{\frac{1}{3}} dy$
 - $I_x = 11878.12016 \text{ units}^4$

- (b) About y -axis
 - Use horizontal element and disk method.
 - $dV = \pi x^2 dy$
 - $dV = \pi y^{\frac{2}{3}} dy$
 - $I = \frac{1}{2} \int_V r^2 dV$
 - $I = \frac{1}{2} \int_0^8 y^{\frac{2}{3}} \pi y^{\frac{2}{3}} dy$
 - $I_y = \frac{1}{2} \pi \int_0^8 y^{\frac{4}{3}} dy$
 - $I_y = \frac{192\pi}{7} \text{ units}^4$

Example

The area bounded by $y = x$, $x = 2$, $y = 0$ is revolved about the line $y = 0$, find the moment of inertia at the solid formed with respect to the axis of revolution.

- $I_x = ?$
- Use vertical element and disk method.
- $dV = \pi y^2 dx$
- $dV = \pi x^2 dx$
- $I_x = \frac{1}{2} \int_V r^2 dV$
- $I_x = \frac{1}{2} \int_0^2 x^2 \pi x^2 dx$
- $I_x = \frac{1}{2} \pi \int_0^2 x^4 dx$
- $I_x = \frac{16}{5} \pi$
- Shortcut eq'n
 - Solid of revolution is a cone.
 - $I_x = \frac{\pi}{10} r^4 h$

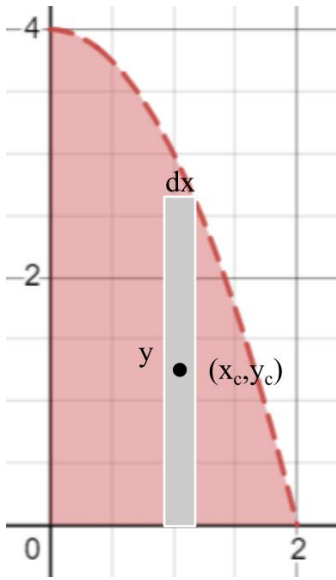
XVI) Centroid of Plane Area

Varignon's Theorem

- $\bar{x} = \frac{\Sigma A \bar{x}}{\Sigma A} = \frac{M_y}{A} = \frac{\int_A x_c dA}{A}$
- $\bar{y} = \frac{\Sigma A \bar{y}}{\Sigma A} = \frac{M_x}{A} = \frac{\int_A y_c dA}{A}$
 - $(\bar{x}, \bar{y})$ – Centroid or Center of Gravity
 - (x_c, y_c) – Centroid of Differential Area

Example

Determine the coordinates of the centroid of the area bounded by the curve $x^2 = -(y - 4)$, the x-axis and the y-axis on the first quadrant.

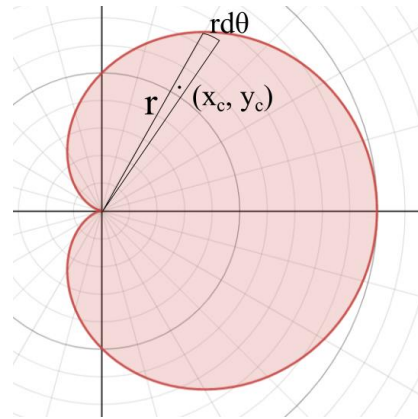


- $x_c = x$
- $y = -x^2 + 4$
- $y_c = \frac{y}{2} = \frac{-x^2 + 4}{2}$
- $dA = y dx$
- $dA = (-x^2 + 4) dx$
- $\bar{x} = \frac{\int_A x_c dA}{A}$
- $\bar{x} = \frac{\int_0^2 x(-x^2 + 4) dx}{\int_0^2 (-x^2 + 4) dx}$
- $\bar{x} = \frac{3}{4}$
- $\bar{y} = \frac{\int_A y_c dA}{A}$
- $\bar{y} = \frac{\int_0^2 (\frac{-x^2 + 4}{2})(-x^2 + 4) dx}{\int_0^2 (-x^2 + 4) dx}$
- $\bar{y} = \frac{8}{5}$
- $(\frac{3}{4}, \frac{8}{5})$

Example

Find the centroid of the area enclosed by the cardioid:

$$r = a(1 + \cos \theta)$$



- $\bar{y} = 0$
 - By symmetry
- $x_c = \frac{2}{3} r \cos \theta = \frac{2}{3} a(1 + \cos \theta)(\cos \theta)$
- $dA = \frac{1}{2} r^2 d\theta = \frac{1}{2} a^2 (1 + \cos \theta)^2 d\theta$
- $\bar{x} = \frac{\int_A x_c dA}{A}$
- $\bar{x} = \frac{\int_0^{2\pi} (\frac{2}{3} a(1 + \cos \theta)(\cos \theta)) (\frac{1}{2} a^2 (1 + \cos \theta)^2) d\theta}{\int_0^{2\pi} (\frac{1}{2} a^2 (1 + \cos \theta)^2) d\theta}$
- $\bar{x} = a \frac{\int_0^{2\pi} (\frac{2}{3} (1 + \cos \theta)(\cos \theta)) (\frac{1}{2} (1 + \cos \theta)^2) d\theta}{\int_0^{2\pi} (\frac{1}{2} (1 + \cos \theta)^2) d\theta}$
- $\bar{x} = \frac{5}{6} a$
- $(\frac{5}{6} a, 0)$

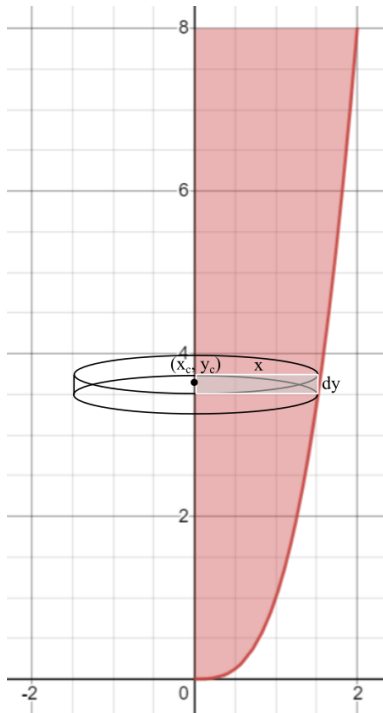
XVII) Centroid of Volume of Solids of Revolution

Formulas

- $\bar{x} = \frac{\Sigma V \bar{x}}{\Sigma V} = \frac{\int_V x_c dV}{V}$
- $\bar{y} = \frac{\Sigma V \bar{y}}{\Sigma V} = \frac{\int_V y_c dV}{V}$
 - (x_c, y_c) is centroid of differential volume dV

Example

Find the centroid of the solid generated when the area bounded by $y = x^3$, $y = 8$ and $x = 0$ is revolved about the y-axis.

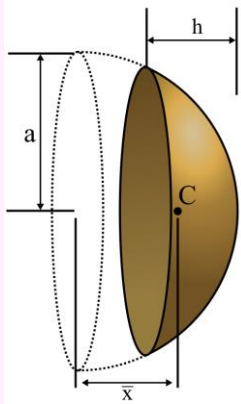


- $x_c = 0; \bar{x} = 0$
- $y_c = \bar{y}$
- $dV = \pi x^2 dy = \pi y^{\frac{2}{3}} dy$
- $\bar{y} = \frac{\int_V y_c dV}{V}$
- $\bar{y} = \frac{\int_0^8 y \pi y^{\frac{2}{3}} dy}{\int_0^8 \pi y^{\frac{2}{3}} dy}$
- $\bar{y} = 5$
- $(0, 5)$

Centroids of Common Solids

Shape	Figure	$\bar{x}$	$\bar{y}$ $(r = a)$
Cylinder		$\frac{1}{2}h$	$\pi r^2 h$
Hemisphere		$\frac{3}{8}r$	$\frac{2}{3}\pi r^3$
Semi-ellipsoid of Revolution		$\frac{3}{8}h$	$\frac{2}{3}\pi r^2 h$
Paraboloid of Revolution		$\frac{1}{3}h$	$\frac{1}{2}\pi r^2 h$
Cone		$\frac{1}{4}h$	$\frac{1}{3}\pi r^2 h$
Pyramid		$\frac{1}{4}h$	$\frac{1}{3}abh$

Spherical Cap
(Spherical Segment with One Base)



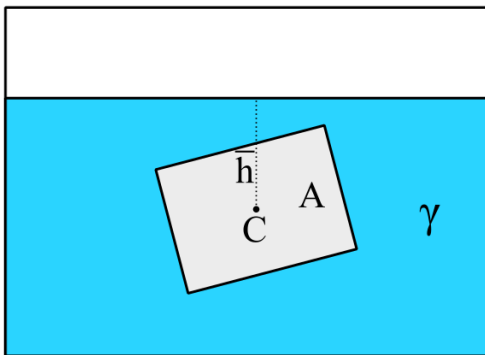
Note: $\bar{x}$ is measured from center of whole sphere

$$\frac{3(2r - h)^2}{4(3r - h)}$$

$$\frac{1}{3}\pi h^2(3r - h)$$

XVIII) Common Application of Centroids in Fluid Mechanics

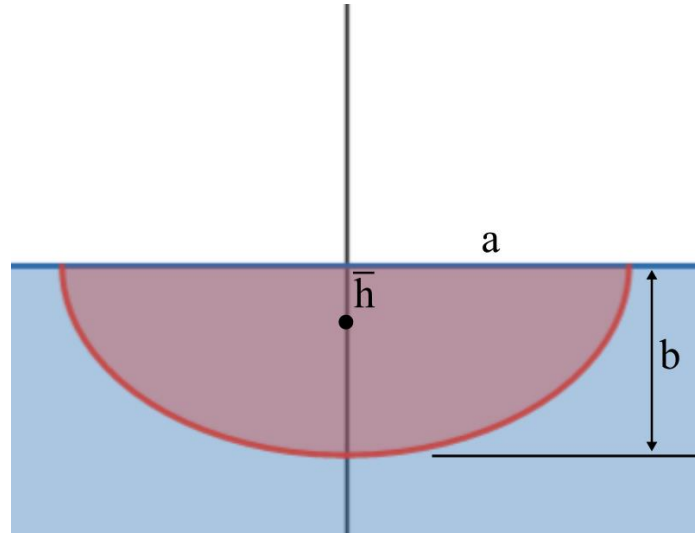
Hydrostatic Force on Plane Surface



- Aka liquid pressure
- $F = \gamma A \bar{h}$
 - F is the hydrostatic force
 - $\bar{h}$ is the vertical distance of the centroid of submerged area from the liquid surface
 - γ is the specific weight of liquid
 - $\gamma_{\text{water}} = 9.81 \text{ kN/m}^3 = 62.4 \text{ lb/ft}^3$

Example

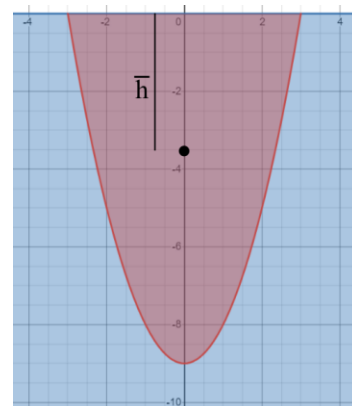
Find the pressure on the submerged vertical lower half of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with the x-axis on the surface of the water.



- $\bar{h} = \frac{4b}{3\pi}$
- $A = \frac{1}{2}\pi ab$
- $F = \gamma A \bar{h}$
- $F = \gamma \cdot \frac{1}{2}\pi ab \cdot \frac{4b}{3\pi}$
- $F = \frac{2}{3}\gamma ab^2$

Example

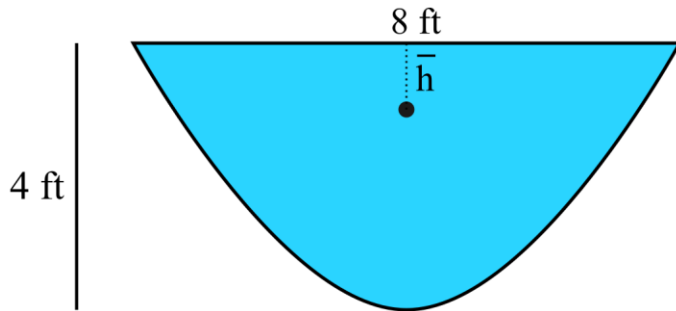
Find the pressure on the submerged vertical area bounded by the parabola $x^2 = y + 9$, and the x-axis, with the x-axis being on the water surface.



- $\bar{h} = \frac{2}{5}h = \frac{2}{5}(9) = \frac{18}{5}$
- $A = \frac{2}{3}bh = \frac{2}{3}(6)(9) = 36$
- $F = \gamma A \bar{h}$
- $F = \frac{648}{5}\gamma$

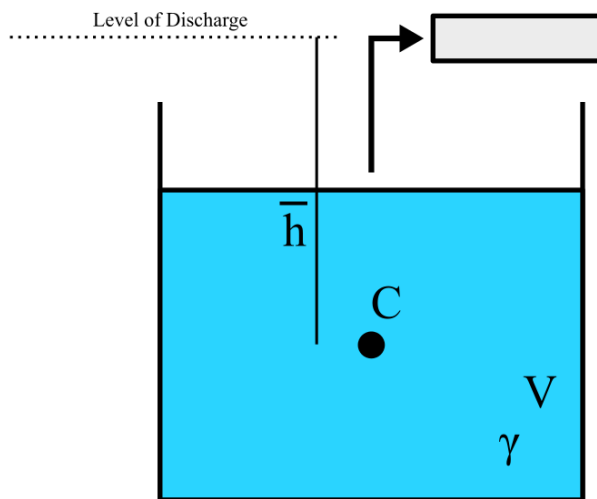
Example

The cross-section of a trough is a parabolic segment 8 ft wide and 4 ft deep. If the trough is filled by liquid weighing 45 lb/ft³, find the total force in Newton at one end.



- Considering one end...
- $A = \frac{2}{3}(4)(8) = \frac{64}{3} \text{ sq. ft.}$
- $\bar{h} = \frac{2}{5}(4) = \frac{8}{5} = \text{ft.}$
- $\gamma = 45 \text{ lb/ft}^3$
- $F = \gamma Ah$
- $F = 1536 \text{ lb} \times \frac{9.81 \text{ N}}{2.204 \text{ lb}} \approx 8 = 6837 \text{ N}$

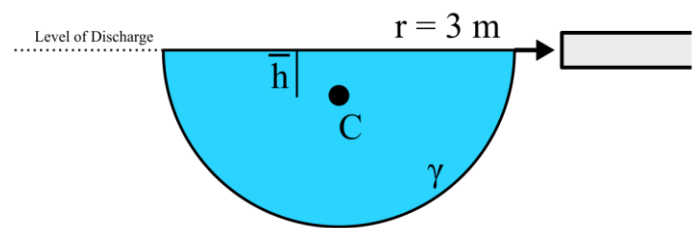
Work Done in Pumping Liquid to LOD



- LOD – “Level of Discharge”
- $W = \gamma V \bar{h}$
 - W is work done in pumping
 - $\bar{h}$ is vertical distance of centroid to level of discharge
 - V is volume of liquid
 - γ is specific weight of liquid
 - $\gamma = s\gamma_{\text{water}}$
 - s is specific weight of liquid

Example

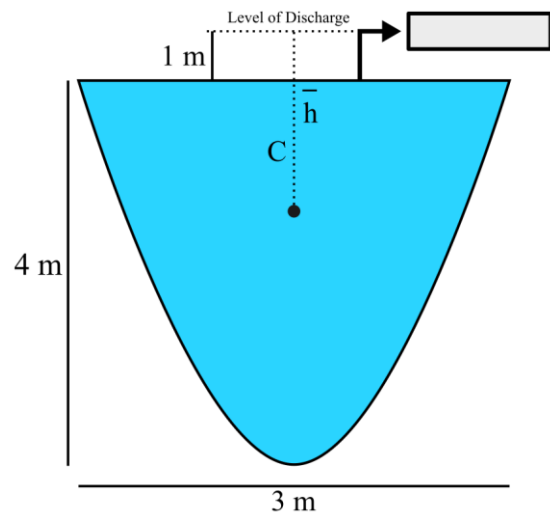
The inner surface of a tank is in the form of hemisphere of radius 3 m with the diametral plane on top. Determine the total work done in pumping the water to the top of the tank. For water, $\gamma = 9.802 \text{ kN/m}^3$.



- $\bar{h} = \frac{3}{8}r = \frac{3}{8}(3) = \frac{9}{8}$
- $V = \frac{2}{3}\pi r^3 = 18\pi$
- $\gamma = 9.802$
- $W = \gamma V \bar{h} = 623.58 \text{ kN} \cdot \text{m}$
- **$W = 623.58 \text{ kJ}$**

Example

A canal is 10 m long and with parabolic end sections which is 3 m wide on top and 4 m deep and is full of water. Find the work done in pumping the water to a level which is 1 m from the top.



- $b = 3 \text{ m}; h = 4 \text{ m}; L = 10 \text{ m}$
- $\bar{h} = 1 + \frac{2}{5}(4) = \frac{13}{5}$
- $V = A_b L = \frac{2}{3}bhL = \frac{2}{3}(3)(4)(10) = 80$
- $\gamma = 9.81 \frac{\text{kN}}{\text{m}^3}$
- $W = \gamma V \bar{h}$
- $W = (9.81)(80)\left(\frac{13}{5}\right)$
- **$W = 2040.48 \text{ kJ}$**

XIX) The Average Value Principle

Average Value

- $y_{ave} = \frac{\int_a^b f(x) dx}{b-a} = \frac{\text{Area}}{\text{Base Width}}$
 - Base width is the length of integration

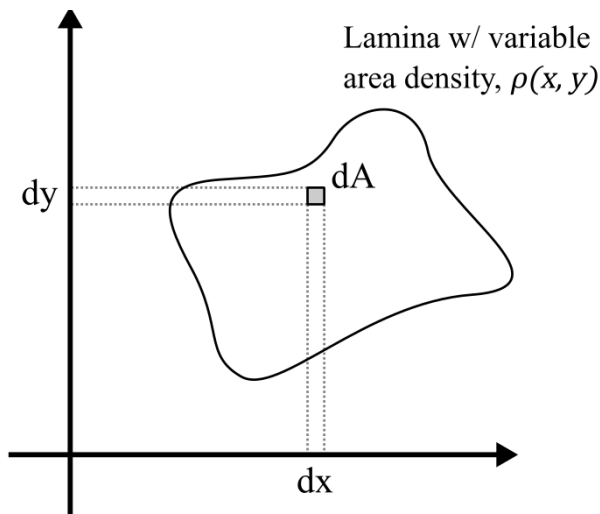
Example

Find the average value of the volume of sphere if its radius changes from 3 cm to 6 cm.

- $V = \frac{4}{3}\pi r^3$
- $V_{ave} = \frac{\int_3^6 \left(\frac{4}{3}\pi r^3\right) dr}{6-3}$
- $V_{ave} = 165\pi \text{ cm}^3$

XX) Some Applications of Double Integral

Mass of Lamina



- $m = \iint_A \rho dA$
- $dA = dydx = dxdy$
 - Outer differential will have constant outer limits.
 - Inner differential can have limits having variables of the outer differentials.

Example

Find the mass of lamina in a given region and density function,

$$D(x, y): 0 \leq x \leq \frac{\pi}{2}, 0 \leq y \leq \cos x$$

$$\rho = 7x$$

- Since x has constant limits, our outer differential is dx with outer limits 0 to $\frac{\pi}{2}$.
- $dA = dydx$
- $m = \iint_A \rho dA$
- $m = \int_0^{\pi/2} \int_0^{\cos x} 7x dy dx$
- $m = \int_0^{\pi/2} [7xy]_{y=0}^{y=\cos x} dx$
- $m = \int_0^{\pi/2} 7x \cos x dx$
- $m = 4$

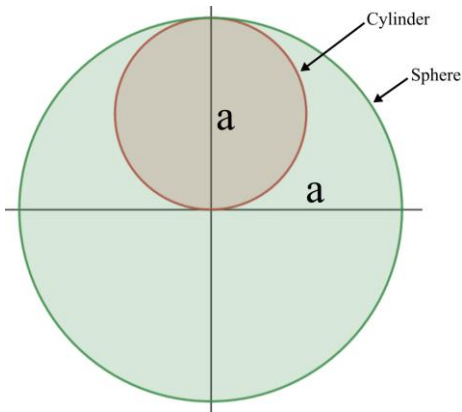
Surface Area

- Surface Area bounded by the function $z = f(x, y)$
- $S = \iint_S \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$
 - dx and dy can be interchanged, the outer differential should be the one with constant limits.

Example

Find the area above the xy -plane of that portion of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ intercepted by the cylinder $x^2 + y^2 - ax = 0$

- $z = \sqrt{a^2 - x^2 - y^2}$
- $\frac{\partial z}{\partial x} = \frac{-2x}{2\sqrt{a^2 - x^2 - y^2}}$
- $\left(\frac{\partial z}{\partial x}\right)^2 = \frac{x^2}{a^2 - x^2 - y^2}$
- $\frac{\partial z}{\partial y} = \frac{-2y}{2\sqrt{a^2 - x^2 - y^2}}$
- $\left(\frac{\partial z}{\partial y}\right)^2 = \frac{y^2}{a^2 - x^2 - y^2}$
- $1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$
 - $= 1 + \frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2}$
 - $= \frac{a^2 - x^2 - y^2 + x^2 + y^2}{a^2 - x^2 - y^2}$
 - $= \frac{a^2}{a^2 - x^2 - y^2}$
- $S = \iint_S \frac{a}{\sqrt{a^2 - x^2 - y^2}} dxdy$
- Convert to polar coordinates
- $dxdy = r dr d\theta$
- $x^2 + y^2 = r$



- Getting the limits
 - From cylinder...
 - $x^2 + y^2 - ax = 0$
 - $x^2 + y^2 = ax$
 - $r^2 = ar \sin \theta$
 - $r = a \sin \theta$
 - $0 \leq r \leq a \sin \theta$
 - How so? Referring to the cylinder in the figure as seen in the xy-plane. When $\theta = 0$, r goes from 0 to 0. When $\theta = 90^\circ$ or π , r goes from 0 to $2a$.
 - θ goes from 0 to π to go over the whole cross-sectional circle of the cylinder.
- $S = \int_0^\pi \int_0^{a \sin \theta} \frac{ar}{\sqrt{a^2 - r^2}} dr d\theta$
- For the inner integral:
 - $u = a^2 - r^2$
 - $du = -2rdr \rightarrow rdr = -\frac{du}{2}$
 - $\int_0^{a \sin \theta} \frac{ar}{\sqrt{a^2 - r^2}} dr = -\frac{a}{2} \int \frac{du}{\sqrt{u}}$
 - $= -\frac{a}{2} \frac{u^{\frac{1}{2}}}{\frac{1}{2}} = -a\sqrt{u} + C$
 - $= -a\sqrt{a^2 - r^2} + C$
- $S = -a \int_0^\pi [\sqrt{a^2 - r^2}]_{r=0}^{r=a \sin \theta} d\theta$
- $S = -a \int_0^\pi (\sqrt{a^2 - a^2 \sin^2 \theta} - \sqrt{a^2}) d\theta$
- $S = -a \int_0^\pi (a|\cos \theta| - a) d\theta$
- $S = -a^2 \int_0^\pi (|\cos \theta| - 1) d\theta$
- $S = -a^2(-1.1415 \dots)$
 - Use calculator for integration.
- $S = -a^2(2 - \pi)$
- $S = (\pi - 2)a^2$

DIFFERENTIAL EQUATIONS

differential equations

A) Definition of Terms

Differential Equation

- Equation with derivatives or differentials
- Ex.:
 - $\frac{dy}{dx} = \frac{y}{x}$
 - Or $ydx - xdy = 0$
 - $y'' - 5y' + 6y = 0$
 - Or $(D^2 - 5D + 6)y = 0$
 - $D \rightarrow \frac{d}{dx}$
 - $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$
 - Laplace's Equation

Ordinary vs Partial DE

- Ordinary DE
 - One independent variable
 - Ex: First and Second Example Above
- Partial DE
 - Two or more independent variable
 - Contains partial derivatives.
 - Ex: Third Example Above

Order of DE

- Order of highest ordered derivative in the DE
- $\frac{dy}{dx} = \frac{y}{x} \rightarrow 1st\ order$
- $(D^2 - 5D + 6)y = 0 \rightarrow 2nd\ order$

Degree of DE

- Integral (whole no.) power of the highest ordered derivative in the DE
- $\frac{dy}{dx} = \frac{y}{x} \rightarrow 1st\ degree$
- $\frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \rightarrow \left(\frac{d^2 y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \rightarrow 2nd\ degree$

General Solution

- The solution to given DE that contains arbitrary constants usually represented as C's.
- No. of constants or C's is order of DE
- Ex:
 - $y = x^2 + 2x + C$
 - $y = C_1 \cos 2x + C_2 \sin 2x$

Particular Solution

- The solution to DE which is evaluated from GS after solving for the values of C's by the use of given or known conditions.

Example

Given the general solution $y = x^2 + 2x + C$ and conditions such that $y = 6$ when $x = 1$, find the particular solution.

- Substitute.
- $6 = (1)^2 + 2(1) + C$
- $C = 6 - 1 - 2$
- $C = 3$
- $y = x^2 + 2x + 3$

I) Derivation of DE

How to Derive a Differential Equation

- Number of Constants = Order of DE
- Find the first order, second order, and so on derivatives of the DE. Stop at order of DE.
- Solve for the constant variables.

Example

Find the differential equation for $y = C_1x + C_2e^x$

- Two arbitrary constants. Indicates differential equation must be second order.
- $y' = C_1 + C_2e^x$
- $y'' = C_2e^x$
- $y' = C_1 + y''$
- $C_1 = y' - y''$
- $y = (y' - y'')x + y''$
- $(x - 1)y'' - xy' + y = 0$

Example

Find the differential equation for the family of parabolas w/ vertices at the origin and their foci on the x-axis.

- Parabolas that open in the left or right direction are considered...
 - Based on description
- $y^2 = \pm ax$
- $y^2 = Cx$
- One arbitrary constant C . Indicates differential equation must be first order.
- $2y y' = C$
- $y^2 = 2yy'x$
- $y' = \frac{y}{2x}$ or $2xdy - ydx = 0$

Example

Find the differential equation for the family of circles with center on the y-axis.

- $x^2 + (y - k)^2 = r^2$
- Two arbitrary constants k and r . Indicates differential equation must be second order.
- $2x + 2(y - k)y' = 0$
- $x + (y - k)y' = 0$
- $y - k = -\frac{x}{y'}$

- $1 + (y - k)y'' + y'y' = 0$
- $1 - \frac{x}{y'}y'' + (y')^2 = 0$
- $xy'' - (y')^3 - y' = 0$

Example

Find the differential equation for the family of lines that passes through a fixed point (h, k) .

- $y - k = m(x - h)$
- One arbitrary constant m . Indicates differential equation must be first-order.
 - k and h is a fixed point. Only m has unknown value.
- $y - k = \frac{dy}{dx}(x - h)$
- $(y - k)dx = (x - h)dy$
- $(y - k)dx - (x - h)dy = 0$

II) Solution to Common 1st Order – 1st Degree DE

Separable DE

- Separate the variables and integrate.
- Model:
 - $f(x)g(y)dx + F(x)G(y)dy = 0$
 - $\frac{f(x)dx}{F(x)} + \frac{G(y)dy}{g(y)} = 0$
 - Multiplying by: $\frac{1}{g(y)F(x)}$
 - $A(x) + B(y) = C$
 - After integrating...

Example

Solve the differential equation.

$$\frac{dx}{dt} = 1 - x - t + xt$$

- $\frac{dx}{dt} = (1 - x)(1 - t)$
- $\frac{dx}{1-x} = (1 - t)dt$
 - $u = 1 - x$
 - $du = -dx$
 - $\int \frac{dx}{1-x} = \int -\frac{du}{u} = -\ln u + C$
 - $v = 1 - t$
 - $dv = -dt$

$$\circ \int (1-t)dt = -\int vdv = -\frac{v^2}{2} = -\frac{1}{2}(1-t)^2 + C$$

- $-\ln(1-x) = -\frac{1}{2}(1-t)^2 + C$
- $2\ln(1-x) - (1-t)^2 = C$
 - Multiplying by -2

Shortcut by Calculator

Solve the differential equation.

$$\frac{dx}{dt} = 1 - x - t + xt$$

- (a) ...
- (b) ...
- (c) ...
- (d) $2\ln(1-x) - (1-t)^2 = C$
 - Set values for x and t .
 - x should be set as less than 1 so $\ln(1-x)$ would not cause an error.
 - Set $x = 0.1, t = 0.2$
 - Solve for $\frac{dx}{dt}$ using the set values...
 - $\frac{dx}{dt} = \frac{18}{25}$
 - Solve for $\frac{dx}{dt}$ in the choices by implicit differentiation.
 - $-\frac{2}{1-x} \frac{dx}{dt} + 2(1-t) = 0$
 - $\frac{dx}{dt} = 0.72 = \frac{18}{25}$
 - **Answer is d.**

Example

Solve the differential equation.

$$\sin x \sin y dx + \cos x \cos y dy = 0$$

- Multiplying by $\frac{1}{\sin y \cos x}$
- $\tan x dx + \cot y dy = 0$
- $\int \tan x dx + \int \cot y dy = 0$
- $\ln|\sec x| - \ln|\csc y| = C$
- $\ln\left|\frac{\sec x}{\csc y}\right| = \ln\left|\frac{1/\cos x}{1/\sin y}\right| = \ln\left|\frac{\sin y}{\cos x}\right| = C$
- Taking e raised to both sides.
- $\left|\frac{\sin y}{\cos x}\right| = e^C = C$
- **$\sin y = C \cos x$**
 - If absolute value is dropped.

Example

Given is $x(y-1)dx - (x+1)dy = 0$ and $x = 4$ when $y = 2$. Find y when $x = 2$.

- Dividing DE by $(y-1)(x+1)$
- $\frac{xdx}{x+1} - \frac{dy}{y-1} = 0$
- $\frac{xdx}{x+1} = \frac{dy}{y-1}$
- $\int_2^4 \frac{xdx}{x+1} = \int_{y_0}^2 \frac{dy}{y-1}$
- Evaluate left side using calculator.
- $1.4892 = \ln|y-1| \Big|_{y=y_0}^{y=2}$
- $1.4892 = \ln 1 - \ln|y_0-1|$
- $1.4892 = -\ln|y_0-1|$
- Shift-solve.
- $y_0 = 1.23$

Exact Differential Equation

$$M(x, y)dx + N(x, y)dy = 0$$

If:

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$M_y = N_x$$

Then:

The differential equation is said to be exact.

To solve:

Set $y = 0$ in M

Set $x = 0$ in N

Example

Solve the given differential equation.

$$2xydx + (2+x^2)dy = 0$$

- Test DE if exact.
- $M = 2xy$
- $M_y = 2x$
- $N = 2 + x^2$
- $N_x = 2x$
- $M_y = N_x, \therefore$ Exact
- Set $y = 0$ in M
- $\int (2+x^2)dy = 0$
- **$(2+x^2)y = C$ or**

- $x^2y + 2y = C$
- If there are choices:
 - Solve $\frac{dy}{dx}$ from DE by setting values for x and y .
 - The solve for $\frac{dy}{dx}$ from choices by implicit differentiation.
 - The following formula can be used.
 - $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

Linear DE

$$\frac{dy}{dx} + p(x)y = g(x)$$

$$y = \frac{\int \mu(x)g(x)dx + C}{\mu(x)}$$

$$\mu(t) = e^{\int p(x)dx}$$

Example

Solve the differential equation.

$$x \frac{dy}{dx} + (1+x)y = 3x$$

- Divide by x
- $\frac{dy}{dx} + \left(\frac{1}{x} + 1\right)y = 3$
- $\mu(x) = e^{\int \left(\frac{1}{x} + 1\right)dx} = e^{\ln x + x} = e^{\ln x} \cdot e^x = xe^x$
- $y = \frac{\int 3xe^x dx + C}{xe^x}$
- Integration by parts in tabular form.

Derive	Integrate
(+) $3x$	e^x
(-) 3	e^x
(+) 0	e^x

- $y = \frac{e^x(3x-3)+C}{xe^x}$ or...
- $yxe^x = e^x(3x-3) + C$
- $e^x(xy - 3x + 3) = C$

Bernoulli's DE

- $\frac{dy}{dx} + p(x)y = q(x)y^n$
- $y^{1-n} = \frac{(1-n)\int \mu(x)q(x)dx + C}{\mu(x)}$
- $\mu(t) = e^{(1-n)\int p(x)dx}$

Example

Solve the differential equation.

$$\frac{dy}{dx} - xy = xy^2$$

- $\mu(x) = e^{(1-2)\int -x dx} = e^{\int x dx} = e^{x^2/2}$
- $y^{1-2} = \frac{(1-2)\int e^{\frac{x^2}{2}}(x)dx}{e^{\frac{x^2}{2}}} + C$
- $y^{-1} = \frac{-\int xe^{x^2/2}dx + C}{e^{x^2/2}}$
- $u = \frac{x^2}{2}$
- $du = xdx$
- $\int xe^{\frac{x^2}{2}}dx = \int e^u du = e^u = e^{\frac{x^2}{2}}$
- $y^{-1} = \frac{-e^{\frac{x^2}{2}} + C}{e^{\frac{x^2}{2}}}$
- Reciprocating both sides...
- $y = \frac{e^{x^2/2}}{C - e^{x^2/2}}$

III) Solutions to Common Higher Order DE

Homogeneous LDE w/ Constant Coefficients

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = 0$$

Right side of equation is **zero**.

- To solve, transform DE to its auxiliary equation or characteristic equation:
 - $D^n = \frac{d^n x}{dx^n} \rightarrow r^n$
 - $a_0 r^n + a_1 r^{n-1} + \dots + a_{n-1} r + a_n = 0$
- General solutions are as follows...
- For real and distinct roots:
 - $y = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \dots$
- For repeated roots: ($r_1 = r_2 = r_3 = \dots$)
 - $y = c_1 e^{r_1 x} + c_2 t e^{r_2 x} + c_3 t^2 e^{r_3 x} + \dots$
- For imaginary roots: $r = j\omega$
 - $y = c_1 \cos \omega x + c_2 \sin \omega x$
- For complex roots: $r = \sigma + j\omega$
 - $y = e^{\sigma x} (c_1 \cos \omega x + c_2 \sin \omega x)$

Example

Solve the differential equation.

$$(D^2 - 5D + 6)y = 0$$

- $r^2 - 5r + 6 = 0$
- $(r - 3)(r - 2) = 0$
- $r = 3, 2$
- $y = c_1 e^{3x} + c_2 e^{2x}$

Example

Solve the differential equation.

$$(D^2 - 4D + 4)y = 0$$

- $r^2 - 4r + 4 = 0$
- $(r - 2)^2 = 0$
- $r = 2, 2$
- $y = c_1 e^{2t} + c_2 t e^{2t}$

1

Example

Solve the differential equation.

$$(D^2 + 10D + 41)y = 0$$

- $r^2 + 10r + 41 = 0$
- $r = -5 \pm j4$
- $y = e^{-5t} (c_1 \cos 4t + c_2 \sin 4t)$

Example

$$(D^4 - 1)y(t) = 0$$

- $r^4 - 1 = 0$
- $(r^2 - 1)(r^2 + 1)$
- $r = \pm 1, \pm j$
- $y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$

Non-Homogeneous LDE with Constant Coefficients

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = f(x)$$

Right side $f(x) \neq 0$

- General solution:
 - $y = y_c + Y_p$
- y_c
 - Complementary solution
 - Homogeneous Solution
 - Can also be y_h
 - To solve, set $f(x) = 0$ and solve like a homogeneous LDE with constant coefficients.
- Y_p
 - Particular solution
 - To solve, do:
 - Method of undetermined coefficients
 - Variation of Parameter – Wronskian

Method of Undetermined Coefficients

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = f(t)$$

$f(x)$	Y_p
$ae^{\beta t}$	$Ae^{\beta t}$
$a \cos(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
$b \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
$a \cos(\beta t) + b \sin(\beta t)$	$A \cos(\beta t) + B \sin(\beta t)$
n-th degree polynomial	$At^n + Bt^{n-1} + \dots + Yt + Z$

- Derive the Y_p with undetermined coefficient, and then substitute to DE.
- Then solve for undetermined coefficient to determine.

Variation of Parameter

Given DE:

$$(a_0 D^n + a_1 D^{n-1} + \dots + a_{n-1} D + a_n)y = f(t)$$

Given homogeneous solution:

$$y_c = c_1 y_1(t) + c_2 y_2(t) + \dots + c_n y_n(t)$$

Particular solution is solved as follows...

Wronskian Determinant

$$W = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

W_n is Wronskian determinant but with nth column

$$\text{replaced by } \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$Y_P(t) = \sum_{k=1}^n \left[y_k(t) \int \frac{W_k(t)}{W(t)} f(t) dt \right]$$

Example

Solve the differential equation.

$$y'' - 4y' + 3y = \sin x$$

- $r^2 - 4r + 3 = 0$
- $(r - 3)(r - 1) = 0$
- $r = 3, 1$
- $y_c = c_1 e^{3x} + c_2 e^x$
- (Method of undetermined coefficients)
- $Y_P = A \cos x + B \sin x$
- $Y_P' = -A \sin x + B \cos x$
- $Y_P'' = -A \cos x - B \sin x$
- Substitute derivatives to DE.
- $Y_P'' - 4Y_P' + 3Y_P = \sin x$
- Collecting $\cos x$
- $-A - 4B + 3A = 0$
- $2A - 4B = 0$
- $A - 2B = 0 \rightarrow (1)$
- Collecting $\sin x$
- $-B + 4A + 3B = 1$
- $4A + 2B = 1 \rightarrow (2)$
- $(1) + (2)$
- $5A = 1$

- $A = \frac{1}{5}$
- $A - 2B = 0$
- $B = \frac{A}{2} = \frac{1}{10}$
- $Y_P = \frac{1}{5} \cos x + \frac{1}{10} \sin x$
- $y = y_c + Y_P$
- $y = c_1 e^{3x} + c_2 e^x + \frac{1}{5} \cos x + \frac{1}{10} \sin x$

Example

$$y'' - 5y' + 4y = 8e^x$$

- $r^2 - 5r + 4 = 0$
- $(r - 4)(r - 1) = 0$
- $r = 4, 1$
- $y_c = c_1 e^{4x} + c_2 e^x$
- $Y_P = Ae^x$ is wrong because the right side function also has e^x , thus we add a factor x
- $Y_P = Ae^x x$
- $Y_P' = Ae^x(x + 1)$
- $Y_P'' = Ae^x(x + 2)$
- $Y_P'' - 5Y_P' + 4Y_P = 8e^x$
- Collecting e^x
- $A(x + 2) - 5A(x + 1) + 4Ax = 8$
- Collecting no x
- $2A - 5A = 8$
- $-3A = 8$
- $A = -\frac{8}{3}$
- $Y_P = -\frac{8}{3} x e^x$
- $y = y_c + Y_P$
- $y = c_1 e^{4x} + c_2 e^x - \frac{8}{3} x e^x$

Example

Solve the differential equation.

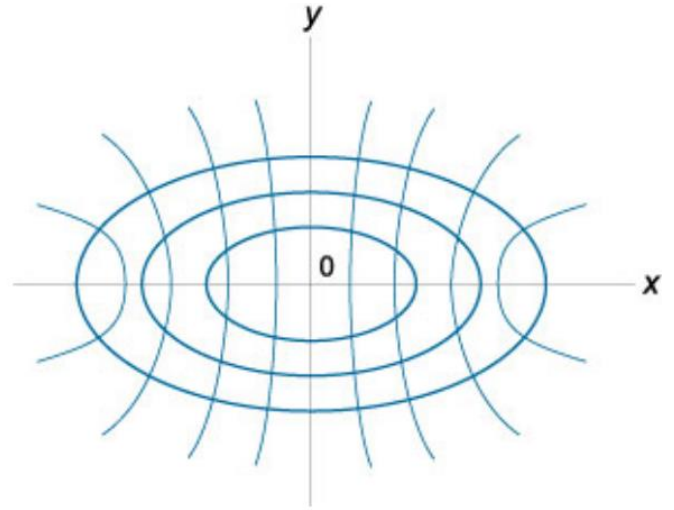
$$y'' + y = \sec x$$

- $r^2 + 1 = 0$
- $r = \pm j$
- $y_c = c_1 \cos x + c_2 \sin x$
- $y_1 = \cos x$
- $y_2 = \sin x$
- By Variation of Parameter

- $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x - (-\sin^2 x)$
- $W = 1$
- $W_1 = \begin{vmatrix} 0 & \sin x \\ 1 & \cos x \end{vmatrix} = -\sin x$
- $W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & 1 \end{vmatrix} = \cos x$
- $f(x) = \sec x$
- $Y_p = y_1 \int \frac{W_1}{W} f(x) dx + y_2 \int \frac{W_2}{W} f(x) dx$
- $y = -\cos x \int \frac{\sin x}{1} \sec x dx + \sin x \int \frac{\cos x}{1} \sec x dx$
- $\int \sin x \sec x dx = \int \frac{\sin x}{\cos x} dx = \int \tan x dx$
- $\int \sin x \sec x = \ln|\sec x|$
- $\int \cos x \sec x = \int \frac{\cos x}{\cos x} dx = \int dx = x$
- $Y_p = -\ln|\sec x| \cos x + x \sin x$
- $Y_p = \ln|\cos x| \cos x + x \sin x$
- $y = y_c + Y_p$
- **$y = c_1 \cos x + c_2 \sin x + \ln|\cos x| \cos x + x \sin x$**
- Note:
 - For this problem, Y_p can be solved by the derived formula:
 - $Y_p = \int_a^x f(\beta) \sin(x - \beta) d\beta + F(a)$
 - $Y_p = F(x) - F(a) + F(a)$
 - $Y_p = F(x)$
 - Just integrate, then substitute $\beta = x$

IV) Common Applications of DE

Orthogonal Trajectories



- Is the family of curves that intersect a given family of curves at right angle.
- Ex:
 - Electrostatics – Lines of Force and Lines of Constant Potential
 - Hydrodynamics – Streamlines and the Lines of Constant Velocity

$$\nabla f(x, y) \cdot \nabla g(x, y) = 0$$

$$\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right) = 0$$

$$\frac{\partial f}{\partial x} \frac{\partial g}{\partial x} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial y} = 0$$

Formulas

- Given Lines R at G defined at some coordinates
- In general:
 - $m_R = -\frac{1}{m_G}$
- In Rectangular CS:
 - $\left(\frac{dy}{dx} \right)_R = -\frac{1}{\left(\frac{dy}{dx} \right)_G}$
 - DE for RCS
- In Polar CS:
 - $\left(r \frac{d\theta}{dr} \right)_R = -\frac{1}{\left(r \frac{d\theta}{dr} \right)_G}$
 - DE for PCS

Example

Find the orthogonal trajectories to the family of parabolas:

$$y^2 = 2x + C$$

- $2y y' = 2$
- $y' = \frac{1}{y}$
- Taking the negative reciprocal of the right side...
- $y' = -y$
- $\frac{dy}{dx} = -y$
- $\frac{dy}{y} = -dx$
- $\ln|y| = -x + C$
- $|y| = e^{-x} e^C$
- $|y| = C e^{-x}$
- Dropping the absolute value...
- $y = C e^{-x}$

Example

Find the orthogonal trajectories to the family of circles:

$$r = a \sin \theta$$

- $r' = a \cos \theta = \frac{dr}{d\theta}$
- $\frac{d\theta}{dr} = \frac{1}{a \cos \theta}$
- $r \frac{d\theta}{dr} = \frac{r}{a \cos \theta} = \frac{a \sin \theta}{a \cos \theta} = \tan \theta$
- Taking the negative reciprocal of the right side...
- $r \frac{dr}{dr} = -\cot \theta$
- $-\tan \theta d\theta = \frac{dr}{r}$
- $-\ln|\sec \theta| = \ln|\cos \theta| = \ln|r| + C$
- $\cos \theta = e^{\ln|r|} e^C$
- $\cos \theta = rC$
- $r = \frac{1}{C} \cos \theta$
- $r = C \cos \theta$

Exponential Growth and Decay

- Or **Exponential Law of Change**
 - The rate of change of a quantity is proportional to the amount present anytime
- DE
 - $\frac{dx}{dt} = kx$
 - $k =$
constant of proportionality
or rate constant
- General Solution:
 - $\ln \left| \frac{x}{x_0} \right| = kt$
 - x_0 amount at $t = 0$
 - x amount at time t
- Applies to:
 - Population Growth
 - Radioactivity – Half-life
 - Common problem is “carbon dating”
 - Half life of carbon-14 is 5700 years.
 - At half-life
 - $x = 0.5x_0$
 - Continuous Compounding of interest
 - Electrical Energy “Power” Forecasting
 - Culture of Bacteria and etc.

Example

Radium decomposed at the rate proportional to the amount present. In 100 years, 100 g becomes 96 g. What is the percentage remaining after 100 years.

- $\frac{dx}{dt} = kx$
- $\frac{dx}{x} = k dt$
- $\ln \left| \frac{x}{x_0} \right| = kt$
- $k = \frac{\ln|x/x_0|}{t} = \frac{\ln|96/100|}{100} = \frac{\ln|0.96|}{100}$
- $P = \frac{x}{x_0} \times 100 = ?$
- $\ln \left| \frac{P}{100} \right| = \frac{\ln|0.96|}{100} (200)$
- $P = 92.16\%$

- (Alternative Solution)
- $\ln \left| \frac{x}{x_o} \right| = kt$
- $\frac{x}{x_o} = e^{kt}$
- $P = \frac{x}{x_o} \times 100$
- $\frac{P}{100} = e^{kt}$
- $P = 100e^{kt}$
- We can use exponential regression model Ae^{Bx} in a calculator. Input first two rows.

$x \rightarrow t$	$y \rightarrow P$
0	100
100	96
200	?

$$200\hat{y} = 92.16$$

- **$P = 92.16\%$**

Example

Carbon extracted from an ancient skull contained only one-sixth as much carbon-14 as carbon extracted from the present day bone. How old is the skull? (Half-life of carbon-14 is 5700 yrs.)

- $\frac{dx}{dt} = kx$
- $\frac{dx}{x} = kdt$
- $\ln \left| \frac{x}{x_o} \right| = kt$
- $\ln|0.5| = k(5700)$
- $k = \frac{\ln|0.5|}{5700}$
- $t = \frac{\ln|x/x_o|}{k} = \ln \left| \frac{1}{6} \right| \cdot \frac{5700}{\ln|0.5|}$
- **$t = 14734 \text{ yrs}$**
- (Alternative Method)
- $\ln \left| \frac{x}{x_o} \right| = kt$
- $\frac{x}{x_o} = e^{kt}$
- We can use exponential regression model Ae^{Bx} in a calculator. Input first two rows.

$x \rightarrow t \text{ (time)}$	$y \rightarrow \frac{x}{x_o} \text{ (amt of C)}$
0	1
5700	0.5

?	$\frac{1}{6}$
---	---------------

$$\frac{1}{6}\hat{x} = 14734$$

- **$t = 14734 \text{ yrs}$**

Example

Upon the birth of their 1st child a couple deposited \$5000 on the account that pays 8% interest compounded continuously.

- $\frac{dS}{dt} = rS$
- $\frac{dS}{S} = rdt$
- $\ln|S/S_o| = rt$
- $\frac{S}{S_o} = e^{rt}$
- $S = S_o e^{rt}$
- $S = 5000e^{0.08(18)}$
- **$S = \$21103.48$**

Example

In a certain country, the energy consumption is expected to double in 10 yrs. Find the average per unit growth rate.

- $P = P_o e^{kt}$
- $2P_o = P_o e^{k(10)}$
- $2 = e^{10k}$
- **$k = 0.0693 \text{ or } 6.93\%$**

Example

The intensity I of light at a depth of x meters below the surface of the lake satisfies the differential equation $\frac{dI}{dx} = -1.4I$. At what depth is the intensity half of the intensity I_o at the surface where $x = 0$.

- $\frac{dI}{dx} = -1.4I$
- $\ln \left| \frac{I}{I_o} \right| = -1.4x$
- $x = \frac{\ln|I/I_o|}{-1.4} = \frac{\ln|1/2|}{-1.4}$
- **$x = 0.495 \text{ m}$**

Newton's Law of Cooling

$$\frac{dT}{dt} = k(T - T_m)$$

$$\ln \left| \frac{T - T_m}{T_o - T_m} \right| = kt$$

$$T - T_m = (T_o - T_m)e^{kt}$$

- T = temperature of body at time t
- k = rate constant
- T_o = initial temperature
- T_m = surrounding temperature

Example

In a room temperature of 25°C, a steel ball at 120°C cools down to 80°C in 20 minutes. Find the temperature after one-half hour.

- $\ln \left| \frac{T - T_m}{T_o - T_m} \right| = kt$
- $\ln \left| \frac{80 - 25}{120 - 25} \right| = k(20)$
- $k = \frac{\ln \left| \frac{11}{19} \right|}{20}$
- $T - T_m = (T_o - T_m)e^{kt}$
- $T = (120 - 25)e^{\frac{\ln \left| \frac{11}{19} \right|}{20} \cdot 30} + 25$
- $T = 66.85^\circ\text{C}$
- (Alternative Method)
- $T - T_m = (T_o - T_m)e^{kt} \rightarrow y = Ae^{Bx}$

$x \rightarrow t$	$y \rightarrow T - T_m$
0	120 - 25
20	80 - 25
30	?

$$30\hat{y} + 25 = 66.85^\circ\text{C}$$

- $T = 66.85^\circ\text{C}$

Example

Just before midday, the body of an apparent homicide victim is found in a room that is kept at a constant temperature of 70°F. At 12 noon, the temperature of the body is 80°F and at 1PM, it is 75°F. Assume that the temperature of the body at the time of death was 98.6°F and it is cooled in accordance with Newton's Law. What was the time of death?

- $T - T_m = (T_o - T_m)e^{kt} \rightarrow y = Ae^{Bx}$

$x \rightarrow t$	$y \rightarrow T - T_m$
?	98.6 - 70 = 28.6
12NN → 12	80 - 70 = 10
1PM → 13	75 - 70 = 5

$$(98.6 - 70)\hat{x} = 10.48 = 10:29:2.35$$

- **10:29:2.35AM**

The Logistic Equation

$$\frac{dP}{dt} = kP(M - P)$$

M = capacity

- Birth rate and death rate population model
 - $\frac{dP}{dt} = (\beta - \delta)P$
 - β = birth rate
 - δ = death rate
- Common Problems:
 - Spreading Virus
 - Spreading Rumor, and others

Example

The rate of virus spread is jointly proportional to those infected and uninfected people. If there are 5000 inhabitants at the start; after a week, there are 160 infected; after 2 weeks, there are 1200 people infected; how many days will there be for 1500 people to be infected?

t	P
7	160
14	1200
?	1500

- $\frac{dP}{dt} = kP(M - P)$
- $\frac{dP}{P(M - P)} = kdt$
- $\int_{160}^{1200} \frac{dP}{P(5000 - P)} = \int_7^{14} kdt = 7k$

- $k = \frac{1}{7} \int_{160}^{1200} \frac{dP}{P(5000-P)}$
- $k = 6.4480 \times 10^{-5} \rightarrow (C)$
- $\int_{160}^{1500} \frac{dP}{P(5000-P)} = \int_7^t k dt = (t-7)k$
- $t = \frac{1}{k} \int_{160}^{1500} \frac{dP}{P(5000-P)} + 7$
- Use stored value in (C) for k
- **$t = 14.947$ days ≈ 15 days**

Example

The time rate of change of rabbit population P is proportional to the square root of P . At time $t = 0$ (months) the population numbers 100 rabbits and is increasing at the rate of 20 rabbits per month. How many rabbits will there be one year later?

t	P
0	100
12	?

- $\frac{dP}{dt} = k\sqrt{P}$
- $20 = k\sqrt{100}$
- $k = 2$
- $\frac{dP}{dt} = 2\sqrt{P}$
- $\frac{dP}{2\sqrt{P}} = dt$
- $\sqrt{P}|_{P=100}^{P_x} = t|_{t=0}^{t=12}$
- $\sqrt{P_x} - 10 = 12$
- $P_x = (12 + 10)^2$
- **$P_x = 484$ rabbits**

Example

Consider an animal population $P(t)$ with constant death rate $\delta = 0.01$ and birth rate β proportional to P . Suppose that $P(0) = 200, P'(0) = 2$:

- When is $P = 1000$?
- When does doomsday occur?

t	P	P'
0	200	2
(a) ?	1000	
(b) ?	∞	

- $\frac{dP}{dt} = (\beta - \delta)P$
- $\beta \propto P \rightarrow \beta = kP$
- $\delta = 0.01$
- $\frac{dP}{dt} = (kP - 0.01)P$
- $P(0) = 200, P'(0) = 2$
- $2 = (200k - 0.01)200$
- $k = 0.0001$
- $\frac{dP}{dt} = (0.0001P - 0.01)P$
- $\frac{dP}{(0.0001P - 0.01)P} = dt$
- $\int_{200}^{1000} \frac{dP}{(0.0001P - 0.01)P} = \int_0^{t_x} dt$
- **$t_x = 58.78$**
- $\int_{200}^{\infty} \frac{dP}{(0.0001P - 0.01)P} = \int_0^{t_y} dt$
- Approximating ∞ in calculator as 999999
- **$t_y = 69.3$**

Acceleration

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

$$dt = \frac{dv}{a} = \frac{ds}{v} \rightarrow a = v \frac{dv}{ds}$$

Newton's Second Law of Motion

$$\Sigma F = ma$$

Example

An arrow is shot straight upward from the ground with an initial velocity of 160 ft/s. It experiences both the deceleration of gravity and deceleration $\frac{v^2}{800}$ due to air resistance. How high in the air does it go? Use $g = 32 \text{ ft/s}^2$.

- $a = \frac{dv}{dt} = v \frac{dv}{ds}$
- $-g - \frac{v^2}{800} = v \frac{dv}{ds}$
- $-32 - \frac{v^2}{800} = -\frac{25600+v^2}{800} = v \frac{dv}{ds}$
- $-\frac{800v dv}{v^2+25600} = ds$
- $-\int_{160}^0 \frac{800v dv}{v^2+25600} = \int_0^h ds$
- $h = 277.26 \text{ ft}$

Example

An object falls from rest. If the velocity of the object before it reached the ground is represented by the differential equation $\frac{dv}{dt}$ plus v per 10 equals 32, ft/s². Find the velocity of the object in ft/s after 1 sec.

- $\frac{dv}{dt} + \frac{v}{10} = 32$
- $\mu(t) = e^{\int \frac{dt}{10}} = e^{\frac{t}{10}}$
- $v(t) = \frac{\int e^{\frac{t}{10}} \cdot 32 dt + C}{e^{\frac{t}{10}}} = \frac{32 \cdot 10 e^{\frac{t}{10}} + C}{e^{\frac{t}{10}}} = 320 + C e^{-\frac{t}{10}}$
- $v(0) = 0$
- $0 = 320 + C$
- $C = -320$
- $v(1) = ?$
- $v(1) = 320 - 320e^{-\frac{1}{10}}$
- $v(1) = 30.45 \text{ ft/s}$

Example

A body falls from rest. If the resistance of the air is proportional to the speed and the limiting speed is 160 ft/s, find the speed in ft/s at the end of 5 sec.

- $R = kv$
- Sign: negative for downward direction, positive for upward direction
- $\Sigma F = ma$
- $W - R = ma$
- $mg - kv = ma$
- Limiting speed is 160 ft/s daw
 - $a = 0, v = 160$
- $mg - 160k = 0$
- $k = \frac{mg}{160}$
- $mg - \frac{mg}{160} \cdot v = ma$
- $g - \frac{gv}{160} = a$
- $g \left(1 - \frac{v}{160}\right) = a$
- $g(160 - v) = 160a = 160 \frac{dv}{dt}$
- $g dt = \frac{160 dv}{160 - v}$
- $\int_0^5 g dt = \int_0^{v_f} \frac{160}{160 - v} dv$
- $-160 \ln|160 - v| \Big|_{v=0}^{v=v_f} = 5(32)$
- $-160 \ln \left| \frac{160 - v_f}{160} \right| = 160$
- $v_f = 101.14 \text{ ft/s}$

Shortcut

- Resistance of air proportional to speed and limiting speed given as v_{max}

$$v = v_{max} \left(1 - e^{-\frac{gt}{v_{max}}} \right)$$

Derivation

- $mg - kv = ma$
- $a = 0, v = v_{max}$
- $mg - v_{max}k = 0$
- $k = \frac{mg}{v_{max}}$
- $mg - \frac{mg}{v_{max}} \cdot v = ma$
- $g - \frac{gv}{v_{max}} = a$
- $g \left(1 - \frac{v}{v_{max}}\right) = a = \frac{dv}{dt}$

$$\begin{aligned}
\circ \quad g \, dt &= \frac{dv}{1 - \frac{v}{v_{\max}}} \\
\circ \quad \int_0^t g \, dt &= \int_0^v \frac{dv}{1 - v/v_{\max}} \\
\circ \quad gt &= -v_{\max} \ln \left| \frac{1 - \frac{v}{v_{\max}}}{1 - \frac{0}{v_{\max}}} \right| \\
\circ \quad -\frac{gt}{v_{\max}} &= \ln \left| 1 - \frac{v}{v_{\max}} \right| \\
\circ \quad e^{-\frac{gt}{v_{\max}}} &= 1 - \frac{v}{v_{\max}} \\
\circ \quad \frac{v}{v_{\max}} &= 1 - e^{-\frac{gt}{v_{\max}}} \\
\circ \quad v &= v_{\max} \left(1 - e^{-\frac{gt}{v_{\max}}} \right)
\end{aligned}$$

Escape Velocity

- It is the velocity of the body projected from the surface of a heavenly body to escape the pull of gravity

$$v_e = \sqrt{2gR} = \sqrt{\frac{2GM}{R}}$$

- g = gravitational acceleration at the surface of heavenly body
- R = radius of heavenly body
- M = mass of heavenly body
- $G = 6.67 \times 10^{-11} \, \text{N} \cdot \frac{\text{m}^2}{\text{kg}^2}$
- For Earth:
 - $R \approx 6400 \, \text{km} = 6.4 \times 10^6 \, \text{m}$
 - $M \approx 6 \times 10^{24} \, \text{kg}$

Example

Find the escape velocity needed by a spacecraft projected from the surface of the Earth so that it may never return.

- $v_e = \sqrt{2gR} = \sqrt{2(9.8)(6.4 \times 10^6)}$
- $v_e = 11200 \, \text{m/s} = 11.2 \, \text{km/s}$

Chemical Solution in Tank

- (Mixture Problem)

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$\frac{dQ}{dt} = q_i c_i - q_o c_o$$

$$c_o = \frac{Q}{v_t}$$

$$\frac{dv}{dt} = q_i - q_o \rightarrow v_t = v_1 + (q_i - q_o)t$$

- q = flow rate or volume per unit time
- c = concentration or salt per volume
- v = volume
- Q = amount of salt or any solute

- $c_i = 0$
- $\frac{dQ}{dt} = q_i c_i - q_o c_o$
- $\frac{dQ}{dt} = -2 \cdot \frac{Q}{t+100}$
- $\frac{dQ}{Q} = -\frac{2 dt}{t+100}$
- $\ln \left| \frac{Q_x}{50} \right| = -2 \ln \left| \frac{60+100}{0+100} \right|$
- $Q_x = 19.53 \text{ kg}$

Example

A tank contains 1000L of solution consisting of 100 kg of salt dissolved in water. Pure water is pumped into the tank at the rate of 5 L/s and the mixture — kept uniform by stirring — is pumped out at the same rate. How long will it be until only 10 kg of salt remains in the tank?

- $q_i = q_o = 5 \text{ L/s}$
- $v_t = v_1 + (q_i - q_o)t = v_1 = 1000 \text{ L}$
- $c_o = \frac{Q}{v_t} = \frac{Q}{1000}$
- $c_i = 0$
- $\frac{dQ}{dt} = q_i c_i - q_o c_o$
- $\frac{dQ}{dt} = -5 \frac{Q}{1000}$
- $\frac{dQ}{Q} = -\frac{1}{200} dt$
- $\ln \left| \frac{10}{100} \right| = -\frac{1}{200} t_x$
- $t_x = 460.52 \text{ s} = 7 \text{ m } 40.52 \text{ s}$

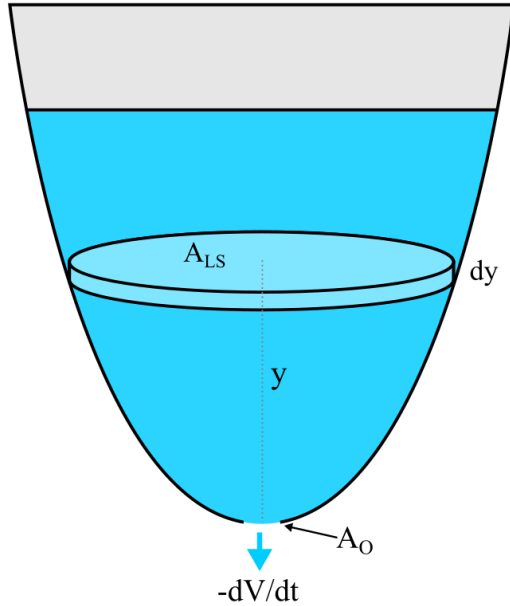
Example

A tank contains 100 L of brine containing 50 kg of salt. Pure water runs into the tank at the rate of 3 L/min. Brine runs out at the rate of 2 L/min. The solution in the tank is kept uniform by stirring. How much salt is in the tank after 1 hour?

- $q_i = 3 \text{ L/min}$
- $q_o = 2 \text{ L/min}$
- $v_t = v_1 + (q_i - q_o)t$
- $v_t = 100 + t$
- $c_o = \frac{Q}{v_t} = \frac{Q}{t+100}$

Orifice w/ Variable or Falling Head

- Fluid Mechanics



$$-\frac{dV}{dt} = CA_O\sqrt{2gy}$$

- $dV = A_{LS}dy$
 - A_{LS} = instantaneous area of liquid surface
- C = coefficient of discharge
 - $C = 1$ for ideal situation
 - $C = 0.6$ for standard orifice
- A_O = area of Orifice
- y = variable depth or head of liquid

Shortcut Formulas

For Tank w/ Constant Cross-Section

$$t = \frac{2A_{LS}(\sqrt{y_o} - \sqrt{y})}{CA_O\sqrt{2g}}$$

t = Time it takes for the depth to change from y_o to y

For hemispherical tank w/ diametral plane on top

$$t = \frac{14\pi r^2}{15CA_O\sqrt{2g}}$$

t = Time it takes to empty the tank through the orifice at the bottom

For Hemispherical Tank w/ diametral plane at the bottom

$$t = \frac{8\pi r^{5/2}}{5CA_O\sqrt{2g}}$$

t = Time it takes to empty the tank through the orifice at the bottom

For Spherical Tank w/ Orifice at the Lowest Point

$$t = \frac{16\sqrt{2}\pi r^2}{15CA_O\sqrt{2g}}$$

t = Time to empty

$$t = \frac{? \pi r^{5/2}}{? CA_O\sqrt{2g}}$$

- For hemispherical tank, half-plane on top
 - $\frac{?}{?} = \frac{14}{15}$
- For hemispherical tank, half-plane at bottom
 - $\frac{?}{?} = \frac{8}{5}$
- For spherical tank
 - $\frac{?}{?} = \frac{16\sqrt{2}}{15}$

Example

A hemispherical bowl with top radius of 4m is full of water. How long will it take to empty the bowl through an 8 cm orifice at the bottom?

- $t = \frac{14}{15} \frac{\pi r^{5/2}}{CA_O\sqrt{2g}}$
- $r = 4$
- $C = 1$
- $A_O = \pi \left(\frac{8 \times 10^{-2}}{2} \right)^2$
- $g = 9.8$
- $t = 4216.37 \text{ s}$
- In calculator, divide by 60 twice, then press ° ' " button.
- $t = 1\text{h } 10\text{m } 16.37\text{s}$

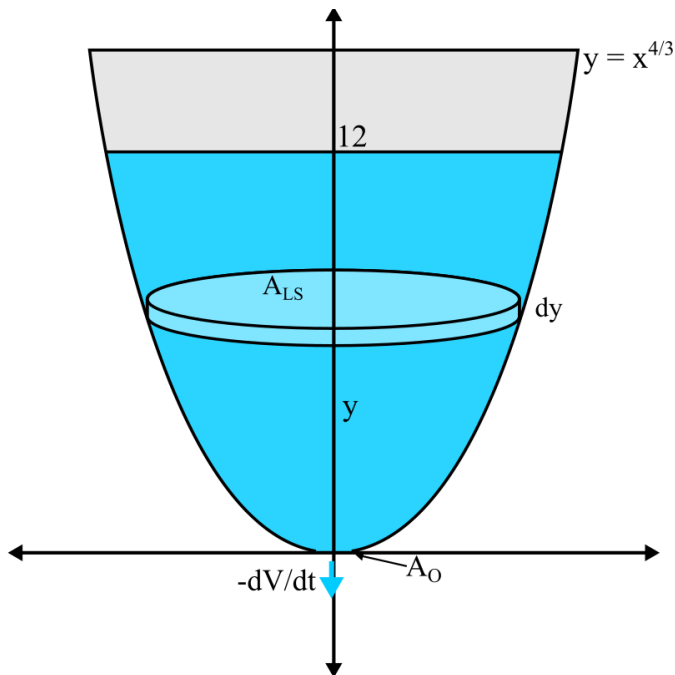
Example

A spherical tank of radius 4 ft is full of gasoline when a circular bottom hole with radius one inch is opened. How long will be required for all the gasoline to drain from the tank?

- $t = \frac{16\sqrt{2}}{15} \frac{\pi r^{5/2}}{CA_o\sqrt{2g}}$
- $r = 4$
- $g = 32$
- $A = \pi \left(\frac{1}{12}\right)^2$
- $C = 1$
- $t = 868.89 \text{ s}$
- Divide by 60 twice and press the degree minutes seconds button.
- **$t = 14\text{m } 28.89\text{s}$**

Example

A water tank has the shape obtained by revolving the curve $y = x^{4/3}$ around the y-axis. A plug at the bottom is removed at 12 noon, when the depth of the water in the tank is 12 ft. At 1 PM, the depth of the water is 6 ft. When will the tank be empty?



- $y = x^{4/3} \rightarrow x = y^{3/4} \rightarrow x^2 = y^{3/2}$
- $A_{LS} = \pi r^2 = \pi x^2 = \pi y^{3/2}$
- $dV = A_{LS} dy = \pi y^{3/2} dy$
- $C = 1$
- $g = 32$

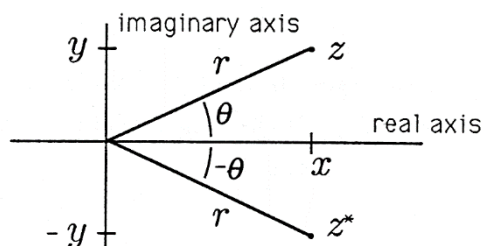
- $-\frac{dV}{dt} = CA_o\sqrt{2gy}$
- $-\frac{\pi y^{3/2} dy}{y^{3/2}} = CA_o\sqrt{2g} dt$
- $-y dy = \frac{CA_o\sqrt{2g}}{\pi} dt = k dt$
- $-\frac{1}{2}(h^2 - h_o^2) = kt$
- Substituting...
 - $h = 6$
 - $h_o = 12$
 - $t = 1$
- $k = 54$
- Substituting...
 - $k = 54$
 - $h_o = 12$
 - $h = 0$
- $t = 1.3 \text{ h}$
- $t = 1\text{h } 20\text{m since } 12 \text{ nn}$
- **1: 20PM**

ADVANCED MATHEMATICS

advanced mathematics

A) Complex Numbers

Complex Numbers



Complex Numbers Forms

$$z = x + jy \quad \text{Rectangular}$$

$$z = r(\cos \theta + j \sin \theta) \quad \text{Trigonometric}$$

$$z = r \angle \theta \quad \text{Shortened Trig.}$$

$$z = r \angle \theta \quad \text{Polar}$$

$$z = re^{j\theta} \quad \text{Exponential or Euler's}$$

Things to Know

$$x = \text{Re}\{z\} \quad \text{Real Part of } Z$$

$$y = \text{Im}\{z\} \quad \text{Imaginary Part of } Z$$

$$i = \sqrt{-1} \quad \text{Unit Imag. Number}$$

$$r = |z| \quad \text{Magnitude / Modulus of } Z$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \arg(z) \quad \text{Argument / Direction Angle of } Z$$

$$\theta = \arctan\left(\frac{y}{x}\right) \pm \phi$$

$$\phi = \pi \text{ if } x < 0$$

Note on Theta

- θ should be in radians in Euler's form.
- θ is positive when measured counter-clockwise from positive x-axis.
- θ is negative when measured clockwise from the positive x-axis.
- When calculator calculates θ , its range is $-180^\circ < \theta < 180^\circ$ or $-\pi < \theta < \pi$.
 - For $\theta > 180^\circ$ it measures counterclockwise instead and outputs negative θ .

Example

Evaluate i^{247}

- $\frac{247}{4} = 61 \text{ r } 3$
- $i^{247} = i^3 = -i$
- $i^{247} = -i$
- Or...
- $i^{247} = (\sqrt{-1})^{247} = (-1)^{247/2}$
- $i^{247} = (-1)^{123.5} = (-1)^{123}(-1)^{0.5}$
- $i^{247} = -i$

Example

Evaluate $(i)^i$

- $i = e^{\frac{\pi}{2}i}$
 - $\frac{\pi}{2}$ is 90 degrees in radians
- $(i)^i = \left(e^{\frac{\pi}{2}i}\right)^i = e^{-\frac{\pi}{2}}$
- $(i)^i = e^{-\pi/2} \approx 0.2079$

Example

Evaluate $i \angle i$

- $z = r \angle \theta = re^{i\theta}$
- $i \angle i = ie^{i \cdot i} = ie^{i^2} = ie^{-1}$
- $i \angle i = \frac{i}{e} \approx 0.3679i$

Conversion from One Form to Another

Find the equivalent of $-3 + 4i$ in exponential form.

$$(a) \ 5e^{\left(\frac{\pi}{2} + \arctan\left(\frac{3}{4}\right)\right)i}$$

$$(b) \ 5e^{\left(\frac{\pi}{2} + \arctan\left(\frac{4}{3}\right)\right)i}$$

$$(c) \ 5e^{\left(\frac{\pi}{2} + \arctan\left(\frac{3}{5}\right)\right)i}$$

$$(d) \ 5e^{\left(\frac{\pi}{2} + \arctan\left(\frac{4}{5}\right)\right)i}$$

- $z = re^{\theta r i}$
- $r = \sqrt{3^2 + 4^2} = 5$
- $\theta_r = \arctan\left(\frac{4}{-3}\right) \pm \pi = 2.214$

- Calculating θ_r of the choices:
 - (a) $\theta_r = 2.214$
- Answer is a.
- Alternatively...
 - Use complex mode in calculator.
 - Input $-3 + 4i$
 - Convert to polar form.
 - The argument is 2.21.
 - Test which argument in the choices will evaluate to the argument value in the phasor form.
 - Answer will be a.**

Basic Operations w/ Complex Numbers

Addition

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2)$$

$$\mathbf{z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)}$$

Subtraction

$$z_1 - z_2 = (x_1 + jy_1) - (x_2 + jy_2)$$

$$\mathbf{z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)}$$

Multiplication

$$z_1 \cdot z_2 = (x_1 + jy_1)(x_2 + jy_2)$$

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$$

$$z_1 \cdot z_2 = r_1 \angle \theta_1 \cdot r_2 \angle \theta_2$$

$$\mathbf{z_1 \cdot z_2 = r_1 r_2 \angle (\theta_1 + \theta_2)}$$

Division

$$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1x_2 + y_1y_2) + j(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2}$$

$$\frac{\mathbf{z_1}}{\mathbf{z_2}} = \frac{\mathbf{r_1 \angle \theta_1}}{\mathbf{r_2 \angle \theta_2}} = \frac{\mathbf{r_1}}{\mathbf{r_2}} \angle (\theta_1 - \theta_2)$$

$$z_1 \cdot z_2 = r_1 r_2 \angle (\theta_1 + \theta_2)$$

Example

If $z_1 = 1 - i$, $z_2 = 2 + 4i$, $z_3 = \sqrt{3} - 2i$, evaluate:

$$\text{Re}\{2z_1^3 + 3z_2^2 - 5z_3^2\}$$

- Store z_1 , z_2 , z_3 in (A), (B), (C)
- Type $2A^3 + 3B^2 - 5C^2$
- The real part is -35.
- $\text{Re}\{2z_1^3 + 3z_2^2 - 5z_3^2\} = -35$**

Theorems for Complex Numbers

$$z = x + jy$$

Negative of z

$$-z = -x - jy$$

Conjugate of z

$$z^* = x - jy$$

Equal Complex Numbers

$$z_1 = x_1 + jy_1 ; z_2 = x_2 + jy_2$$

$$z_1 = z_2$$

$$x_1 = x_2$$

$$y_1 = y_2$$

Example

Find the value of $x + y$ in the complex expression $3 + xi = y + 2i$.

- Equating real and imaginary part of both sides of equations...
- $3 = y$
- $x = 2$
- $x + y = 3 + 2$
- $x + y = 5$**

Example

Find the value of y from the given equation below if x and y are real numbers.

$$\frac{1}{8 + 4i} = \frac{1}{x} + \frac{1}{10 + yi}$$

- Multiply the fractions that are reciprocals of a complex number by the conjugate of the complex number.
- $\frac{1}{8 + 4i} \cdot \frac{8 - 4i}{8 - 4i} = \frac{1}{x} + \frac{1}{10 + yi} \cdot \frac{10 - yi}{10 - yi}$
- $\frac{8 - 4i}{8^2 + 4^2} = \frac{1}{x} + \frac{10 - yi}{10^2 + y^2}$
- Equating real parts:
- $\frac{8}{80} = \frac{1}{10} = \frac{1}{x} + \frac{10}{10^2 + y^2}$
- Equating imaginary parts:
- $-\frac{4}{80} = -\frac{1}{20} = -\frac{y}{10^2 + y^2}$
- $y = 10$**
- $x = 20$**

Example

Find $|e^{iz}|$ if $z = 6e^{i\pi/3}$

- $z = 3 + 3\sqrt{3}i$
 - Utilize complex mode for converting from polar to rectangular form.
- $iz = -3\sqrt{3} + 3i$
- $e^{iz} = e^{-3\sqrt{3}} \cdot e^{3i}$
- $|e^{iz}| = e^{-3\sqrt{3}}$

More Operations w/ Complex Numbers

$$z = r\angle\theta$$

Powers of Complex Numbers

$$z^n = r^n \angle (n\theta)$$

De Moivre's Theorem

$$z^n = r^n \angle ((\theta + 2\pi k)n) \quad k = 0, 1, 2, \dots$$

Example

Find all values of $1^{\sqrt{2}}$

- $\cos \sqrt{2}k\pi + i \sin \sqrt{2}k\pi$
 - $\cos 2\sqrt{2}k\pi + i \sin 2\sqrt{2}k\pi$
 - $\cos(\sqrt{2} + k\pi) + i \sin(\sqrt{2} + k\pi)$
 - $\cos(2\sqrt{2} + k\pi) + i \sin(2\sqrt{2} + k\pi)$
- $1 = 1\angle 0 = 1\angle(0 + 2\pi k) = 1\angle 2\pi k$
 - $1^{\sqrt{2}} = 1^{\sqrt{2}} \angle ((0 + 2\pi k)\sqrt{2})$
 - $1^{\sqrt{2}} = 1^{\sqrt{2}} \angle (2\sqrt{2}\pi k)$
 - $1^{\sqrt{2}} = 1\angle(2\sqrt{2}\pi k)$
 - The argument of cosine and sine must be $2\sqrt{2}\pi k$
 - **Answer is b.**

Roots of Complex Numbers

$$z = r\angle\theta$$

De Moivre's Theorem

$$z^{1/n} = r^{1/n} \angle \left(\frac{\theta + 2\pi k}{n} \right) \quad k = 0, 1, 2, \dots (n-1)$$

$k = 0 \rightarrow$ principal roots

$k = 1, 2, \dots (n-1) \rightarrow$ secondary roots

Note:

- Sum of All Roots = Zero
- Roots represents the vertices of regular polygon with n-sides.

Example

Find all the roots of

$$z^4 + z^2 + 1 = 0$$

- $(z^2)^2 + z^2 + 1 = 0$
- Apply quadratic equation...
- $z^2 = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$
- $z^2 = 1\angle \pm \frac{2\pi}{3}$
- $z = \left(1\angle \frac{2\pi}{3}\right)^{\frac{1}{2}} = 1\angle \left(\frac{\frac{2\pi}{3} + 2\pi k}{2}\right)$
 - $k = 0 \rightarrow z = 1\angle \frac{\pi}{3}$
 - $k = 1 \rightarrow z = 1\angle -\frac{2\pi}{3}$
- $z = \left(1\angle -\frac{2\pi}{3}\right)^{\frac{1}{2}} = 1\angle \left(\frac{-\frac{2\pi}{3} + 2\pi k}{2}\right)$
 - $k = 0 \rightarrow z = 1\angle -\frac{\pi}{3}$
 - $k = 1 \rightarrow z = 1\angle \frac{2\pi}{3}$
- $z = 1\angle \frac{\pi}{3}, 1\angle -\frac{2\pi}{3}, 1\angle -\frac{\pi}{3}, 1\angle \frac{2\pi}{3}$

Logarithmic Properties of Complex No.

$$z = r\angle\theta$$

$$1. \log_b z = \log_b r + (\theta i) \log_b e$$

b = base, real no.

$$2. \log z = \log r + (\theta i) \log e$$

$$\log = \log_{10}$$

$$3. \ln z = \ln r + \theta i$$

$$\ln = \log_e$$

$$4. \log_{z_2} z_1 = \frac{\ln z_1}{\ln z_2}$$

$$\log_b M = \frac{\log_a M}{\log_a b}$$

Deriving First Property

- $z = r\angle\theta$
- $z = re^{j\theta}$
- $\log_b z = \log_b(re^{j\theta})$
- $\log_b z = \log_b r + \log_b e^{j\theta}$
- $\log_b z = \log_b r + j\theta \log_b e$
- Yay!

Example

Evaluate $\log(3 + 4i)$

- $3 + 4i = 5\angle 0.927$
- $\log(5\angle 0.927) = \log 5 + 0.927i \log e$
- **$\log(3 + 4i) = 0.699 + 0.403i$**

Example

Evaluate $\ln(3 - 4i)$

- $3 - 4i = 5\angle -0.927$
- $\ln(5\angle -0.927) = \ln 5 - 0.927i$
- **$\ln(3 - 4i) = 1.609 - 0.927i$**

Example

Evaluate $\ln(-5)$

- $-5 = 5\angle\pi$
- $\ln(5\angle\pi) = \ln 5 + \pi i$
- **$\ln(-5) = 1.609 + 3.142i$**

Example

Evaluate $\log_i(3 + 4i)$

- $\log_i(3 + 4i) = \frac{\ln(3+4i)}{\ln(i)} = \frac{\ln(5\angle 0.927)}{\ln(1\angle \frac{\pi}{2})}$
- $\log_i(3 + 4i) = \frac{\ln 5 + 0.927i}{\ln 1 + \frac{\pi}{2}i}$
- **$\log_i(3 + 4i) = 0.590 - 1.025i$**

Complex Number Raised to a Complex Number

$$(x + yi)^{(a+bi)} = (r\angle\theta)^{(a+bi)} = (r^{a-b\theta})\angle(b\ln r + a\theta)$$

Example

Evaluate $(1 + i)$ raised to $(1 - i)$

- $y = (1 + i)^{(1-i)}$
- $\ln y = (1 - i) \ln(1 + i) = (1 - i) \ln(\sqrt{2}e^{\frac{\pi}{4}i})$
- $\ln y = (1 - i) \cdot (\ln \sqrt{2} + \frac{\pi}{4}i) \rightarrow A$
- Store complex number in (A)
- $y = e^A = e^{Re(A)}\angle Im(A)$
- **$y = 2.808 + 1.318i$**

Example

Find all values of $Re\{(-i)^{(-i)}\}$

- $y = (-i)^{(-i)}$
- $\ln y = -i \ln(-i) = -i \ln\left(1\angle\left(-\frac{\pi}{2} + 2\pi k\right)\right)$
- $\ln y = -i \cdot \left(\ln 1 + \left(-\frac{\pi}{2} + 2\pi k\right)i\right)$
- $\ln y = -\frac{\pi}{2} + 2\pi k$
- **$y = e^{-\frac{\pi}{2} + 2\pi k}$**

TRIG and HYP Functions for Complex Numbers

Trigonometric

$$\sin x = \frac{e^{jx} - e^{-jx}}{j2}$$

$$\cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\tan x = \frac{e^{jx} - e^{-jx}}{j(e^{jx} + e^{-jx})}$$

Hyperbolic

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Trigonometric Functions

1. $\sin(\pm jy) = \pm j \sinh y$
2. $\cos(\pm jy) = \cosh y$
3. $\tan(\pm jy) = \frac{\sin(\pm jy)}{\cos(\pm jy)} = \pm j \tanh y$
4. $\sin(x \pm jy) = \sin x \cosh y \pm j \cos x \sinh y$
5. $\cos(x \pm jy) = \cos x \cosh y \mp j \sin x \sinh y$

Hyperbolic Functions

1. $\sinh(\pm jy) = \pm j \sin y$
2. $\cosh(\pm jy) = \cos y$
3. $\tanh(\pm jy) = \frac{\sinh(\pm jy)}{\cosh(\pm jy)} = \pm j \tan y$
4. $\sinh(x \pm jy) = \sinh x \cos y \pm j \cosh x \sin y$
5. $\cosh(x \pm jy) = \cosh x \cos y \pm j \sinh x \sin y$

Note: Use Radian Unit of Angle!!!

Example

Evaluate $\tan^2(j\frac{\pi}{4})$

- $= \left(\tan \left(j\frac{\pi}{4} \right) \right)^2$
- $= \left(j \tanh \left(\frac{\pi}{4} \right) \right)^2$
- $= -0.4301$

B) The Laplace Transform

Laplace Transform Definition

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = F(s)$$

$$e^{-st} = \text{kernel function of } f(t)$$

$$\mathcal{L}\{f(t)\} = F(s)$$

Calculator Approximation

$$\mathcal{L}\{F(x)\} = \int_0^9 e^{-8x} F(x) dx = ?$$

Calculate the choices at $s = 8$ and choose the choice that is closest to the ?

$f(t)$	$F(s)$
1	$\frac{1}{s}$
k	$\frac{k}{s}$
t^n $n = 0, 1, 2, \dots$	$\frac{n!}{s^{n+1}}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$ $\Gamma(n+1) = \int_0^{\infty} e^{-t} t^n dt$
e^{at}	$\frac{1}{s-a}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh \beta t$	$\frac{s}{s^2 - \beta^2}$
$\sinh \beta t$	$\frac{\beta}{s^2 - \beta^2}$
$e^{at} f(t)$	$F(s-a)$ $F(s) _{s \rightarrow s-a}$
$e^{at} \cos \omega t$	$\frac{s-a}{(s-a)^2 + \omega^2}$
$e^{at} \sin \omega t$	$\frac{\omega}{(s-a)^2 + \omega^2}$

$f(t)$	$F(s)$
$e^{at} \cosh \beta t$	$\frac{s-a}{(s-a)^2 - \beta^2}$
$e^{at} \sinh \beta t$	$\frac{\beta}{(s-a)^2 - \beta^2}$
$t^n f(t)$	$(-1)^n F^{(n)}(s)$
$\int_0^t f(\tau) d\tau$	$\frac{F(s)}{s}$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(3)}(t)$	$s^3 F(s) - s^2 f(0) - sf'(0) - f''(0)$
$\frac{f(t)}{t}$	$\int_s^\infty F(s) ds$

Unit Step Function

$u(t-a)$	$\frac{1}{s} e^{-as}$
$f(t-a)u(t-a)$	$F(s)e^{-as}$

Dirac Delta Function

$\delta(t)$	1
$f(t)\delta(t-a)$	$f(a)e^{-as}$

Periodic Function

$f(t)$ with period T	$\frac{\int_0^T e^{-st} f(t) dt}{1 - e^{-Ts}}$
------------------------	--

Particular Values of the Gamma Function

$$\Gamma(n) = (n-1)! \text{ or } \Gamma(n+1) = n!$$

$$\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n n!} \sqrt{\pi}$$

$$\Gamma(n+1) = n\Gamma(n)$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

- $\int_0^9 e^{-8x} \cos(3x) dx = 0.1096$
- Calculate the choices with the same values of w and s .
- $\frac{s}{s^2+w^2} = \mathbf{0.1096}$

Example

Evaluate $\mathcal{L}\{e^{3t+1}\}$

- $f(t) = e \cdot e^{3t}$
- $F(s) = \frac{e}{s-3}$

Example

Evaluate $\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$

- $f(t) = t^{-\frac{1}{2}}$
- $n = -\frac{1}{2}$
- $F(s) = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{s}}$
- $F(s) = \sqrt{\frac{\pi}{s}}$

Example

Evaluate $\mathcal{L}\{t^{\frac{3}{2}} e^{-4t}\}$

- $f(t) = e^{-4t} t^{\frac{3}{2}}$
- $F(s) = \frac{\Gamma(\frac{3}{2}+1)}{s^{\frac{3}{2}+1}} \Big|_{s \rightarrow s+4}$
- $F(s) = \frac{\Gamma(\frac{1}{2}+2)}{(s+4)^{\frac{5}{2}}}$
- $\Gamma\left(\frac{1}{2} + 2\right) = \frac{(2 \cdot 2)!}{4^2 2!} \sqrt{\pi} = \frac{3}{4} \sqrt{\pi}$
- $F(s) = \frac{3\sqrt{\pi}}{4(s+4)^{\frac{5}{2}}}$
- Alternative evaluation of gamma function
 - Using $\Gamma(n+1) = n\Gamma(n)$
 - $\Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \frac{3}{2} \Gamma\left(\frac{1}{2} + 1\right)$
 - $\Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$
 - $\Gamma\left(\frac{3}{2} + 1\right) = \frac{3}{2} \frac{1}{2} \sqrt{\pi} = \frac{3}{4} \sqrt{\pi}$

Example

Evaluate $\mathcal{L}\{\cos wt\}$

- $F(s) = \frac{s}{s^2+w^2}$
- Pag nabobo ka at nalimutan mo...
- $w = 3$
- $s = 8$

Example

Evaluate $\mathcal{L}\{t \cos at\}$

- $n = 1$
- $f(t) = \cos at$
- $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)$
 - $= -\frac{d}{ds} \left(\frac{s}{s^2 + a^2} \right)$
 - $= -\frac{s^2 + a^2 - s(2s)}{(s^2 + a^2)^2} = -\frac{-s^2 + a^2}{(s^2 + a^2)^2}$
 - $= \frac{s^2 - a^2}{(s^2 + a^2)^2}$

Example

Evaluate at $\mathcal{L}\left\{\int_0^t \cos 2u \, du\right\}$

- $f(t) = \cos 2t$
- $F(s) = \frac{s}{s^2 + 4}$
- $\mathcal{L}\left\{\int_0^t \cos 2u \, du\right\} = \frac{F(s)}{s}$
 - $= \frac{1}{s^2 + 4}$

Example

Evaluate $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

- $f(t) = \sin t$
- $F(s) = \frac{1}{s^2 + 1}$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_0^\infty F(s) \, ds$
 - $= \int_0^\infty \frac{ds}{s^2 + 1}$
 - $= \arctan s \Big|_{s=0}^{s=\infty}$
 - $= \frac{1}{2}\pi - \arctan s$
- Simplifying...
 - $A = \frac{\pi}{2} - \arctan s$
 - $\tan A = \tan\left(\frac{\pi}{2} - \arctan s\right)$
 - $\tan A = \cot(\arctan s)$
 - $\tan A = \frac{1}{\tan(\arctan s)} = \frac{1}{s}$
 - $A = \arctan\left(\frac{1}{s}\right)$
- $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \arctan\left(\frac{1}{s}\right)$

Example

Evaluate the following:

- (1) $\mathcal{L}\left\{\cos\left(t - \frac{\pi}{4}\right)u\left(t - \frac{\pi}{4}\right)\right\}$
- (2) $\mathcal{L}\{t^2 e^{-t} \delta(t - 3)\}$

- Solution for (1)
- $f(t) = \cos t$
- $F(s) = \frac{s}{s^2 + 1}$
- $\mathcal{L}\{f(t - a)u(t - a)\} = F(s)e^{-as}$
 - $= \frac{s}{s^2 + 1} e^{-\frac{\pi}{4}s}$
- Solution for (2)
- $f(t) = t^2 e^{-t}$
- $f(3) = 3^2 e^{-3}$
- $\mathcal{L}\{f(t)\delta(t - a)\} = f(a)e^{-as}$
 - $= 9e^{-(3+3s)}$

C) Use of Unit Step Function in Solvin Laplace Transform of Piecewise Continuous Function

Rewriting a Piecewise Function in terms of u(t)

$$f(t) = \begin{cases} f_1(t) & 0 \leq t < \tau \\ f_2(t) & t \geq \tau \end{cases}$$

$$f(t) = f_1(t) + u(t - \tau)(f_2(t) - f_1(t))$$

Example

Find the Laplace Transform of:

$$f(t) = \begin{cases} 2 & 0 \leq t \leq 3 \\ e^t & t > 3 \end{cases}$$

- $f(t) = 2 + u(t - 3)(e^t - 2)$
- $f(t) = 2 + u(t - 3)(e^3 e^{t-3} - 2)$
- $F(s) = \frac{2}{s} + \left(\frac{e^3}{s-1} - \frac{2}{s}\right)e^{-3s}$

Example

Find the Laplace Transform of:

$$f(t) = \begin{cases} \sin 3t & 0 \leq t < \frac{\pi}{2} \\ 0 & t \geq \frac{\pi}{2} \end{cases}$$

- $f(t) = \sin 3t + u(t - \frac{\pi}{2})(0 - \sin 3t)$

- $f(t) = \sin 3t - u(t-3)(\sin 3t)$
- $f(t) = \sin 3t - u(t-3)\left(\sin\left(3\left(t - \frac{\pi}{2}\right) + \frac{3\pi}{2}\right)\right)$
- $A = 3\left(t - \frac{\pi}{2}\right), B = \frac{3\pi}{2}$
- $\cos(B) = 0$
- $\sin(B) = -1$
- $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- $\sin(A+B) = -\cos\left(3\left(t - \frac{\pi}{2}\right)\right)$
- $f(t) = \sin 3t + u(t-3) \cos\left(3\left(t - \frac{\pi}{2}\right)\right)$
- $F(s) = \frac{3}{s^2+9} + \frac{se^{-\frac{\pi}{2}s}}{s^2+9}$
- $F(s) = \frac{3+se^{-\frac{\pi}{2}s}}{s^2+9}$

Example

Find the Laplace transform of $f(x)$ if $f(x) = 2$ when $0 < x < 3$, and $f(x) = x$ when $x \geq 3$.

- $f(x) = 2 + u(t-3)(x-2)$
- $f(x) = 2 + u(t-3)((x-3) + 1)$
- $F(s) = \frac{2}{s} + \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-3s}$

Example

Evaluate the Laplace transform of the half-wave rectified sine curve represented by:

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi < t < 2\pi \end{cases}$$

- $T = 2\pi$
- $F(s) = \frac{\int_0^T e^{-st} f(t) dt}{1-e^{-Ts}} = \frac{\int_0^\pi e^{-st} \sin t dt}{1-e^{-2\pi s}}$
- Evaluating the integral:

Derive		Integrate
e^{-st}	+	$\sin t$
$-se^{-st}$	-	$-\cos t$
$\int s^2 e^{-st}$	+	$-\sin t$

- $\int e^{-st} \sin t dt = I$
- $I = -e^{-st}(\cos t + s \sin t) - s^2 I$
- $I = -\frac{e^{-st}}{s^2+1}(\cos t + s \sin t)$
- $\int_0^\pi e^{-st} \sin t dt = I|_{t=0}^{t=\pi}$

$$\circ = -\frac{1}{s^2+1}(-e^{-\pi s} - 1)$$

$$\circ = \frac{1+e^{-\pi s}}{s^2+1}$$

$$\bullet F(s) = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})}$$

$$\circ \text{ Since } 1 - e^{-2\pi s} = (1 - e^{-\pi s})(1 + e^{-\pi s})$$

$$\bullet F(s) = \frac{1}{(s^2+1)(1-e^{-\pi s})}$$

• (Alternative method)

• Rewrite $f(t)$

$$\bullet f(t) = \sin t - u(t-\pi)(\sin t)$$

$$\bullet \sin t = \sin((t-\pi) + \pi)$$

$$\bullet A = t - \pi, B = \pi$$

$$\bullet \cos B = -1$$

$$\bullet \sin B = 0$$

$$\bullet \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\bullet \sin(A+B) = -\sin(t-\pi) = \sin t$$

$$\bullet f(t) = \sin t + u(t-\pi)(\sin(t-\pi))$$

$$\bullet \frac{F(s)}{1-e^{-Ts}} = \frac{1}{1-e^{-2\pi s}} \left[\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1} \right]$$

$$\circ = \frac{1+e^{-\pi s}}{(1-e^{-2\pi s})(s^2+1)}$$

$$\circ = \frac{1}{(1-e^{-\pi s})(s^2+1)}$$

Note:

- Laplace transforms can be solved by calculator approximation.

$$\mathcal{L}\{f(x)\} = \int_0^9 e^{-8x} f(x) dx = ?$$

(Calculate the choices at $s = 8$ and choose the choice that is closest to the ?)

D) The Inverse Laplace Transform

Inverse Laplace Transform

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$

$F(s)$	$f(t)$
$\frac{k}{s}$	k
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{(s-a)^n}$	$\frac{e^{at} t^{n-1}}{(n-1)!}$
$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$
$\frac{s}{s^2 - \beta^2}$	$\cosh \beta t$
$\frac{\omega}{s^2 + \omega^2}$	$\sin \omega t$
$\frac{1}{s^2 + \omega^2}$	$\frac{1}{\omega} \sin \omega t$
$\frac{\beta}{s^2 - \beta^2}$	$\sinh \beta t$
$\frac{1}{s^2 - \beta^2}$	$\frac{1}{\beta} \sinh \beta t$
$F(s-a)$	$e^{at} f(t)$
$\frac{s-a}{(s-a)^2 + \omega^2}$	$e^{at} \cos \omega t$
$\frac{\omega}{(s-a)^2 + \omega^2}$	$e^{at} \sin \omega t$
$\frac{1}{(s-a)^2 + \omega^2}$	$\frac{1}{\omega} e^{at} \sin \omega t$
$\frac{s}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega} t \sin \omega t$
$\frac{1}{(s^2 + \omega^2)^2}$	$\frac{1}{2\omega^3} (\sin \omega t - \omega t \cos \omega t)$

Convolution Theorem

$F(s) \cdot G(s)$ $F(s)$ and $G(s)$ are interchangeable	$\int_0^t f(\tau) g(t-\tau) d\tau$
--	------------------------------------

Heaviside Expansion Formulas

$\frac{N(s)}{D(s)}$ $D(s) = 0 \rightarrow s_1, s_2, \dots$ $s_1, s_2, \dots$ must be real and distinct	$\frac{N(s_1)}{D'(s_1)} e^{s_1 t} + \frac{N(s_2)}{D'(s_2)} e^{s_2 t} + \dots$
$\frac{N(s)}{D(s)}$ $D(s) = 0 \rightarrow s_{1,2} = a \pm j\omega$ $\frac{N(s_1)}{D'(s_1)} = \alpha + \beta y$	$e^{at} (2\alpha \cos \omega t - 2\beta \sin \omega t)$

Partial Fraction Expansion or Decomposition

$$\mathcal{L}^{-1}\left\{\frac{N(s)}{D(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{A}{?} + \frac{B}{?} + \frac{C}{?} + \dots\right\}$$

Factors in denominator	Partial Fraction
Linear	
$as + b$	$\frac{A}{as + b}$
Quadratic	
$as^2 + bs + c$	$\frac{As + B}{as^2 + bs + c}$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$

$$\bullet = \frac{1}{2} \sin 2t$$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\}$

$$\bullet = 2e^{-3t}$$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{s}{(s-2)^2+9}\right\}$

- $= \mathcal{L}^{-1}\left\{\frac{(s-2)+2}{(s-2)^2+9}\right\}$
- $= e^{2t}\left(\cos 3t + \frac{2}{3}\sin 3t\right)$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{3}{s} + \frac{s}{s^2-4} - \frac{1}{s^2+9}\right\}$

- $= 3 + \cosh 2t - \frac{1}{3}\sin 3t$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{s^3-4s^2+s+6}\right\}$

- $D(s) = 0$
- $s_{1,2,3} = -1, 3, 2$
- $N(s) = 2s^2 - 4$
- $D'(s) = 3s^2 - 8s + 1$

s	$\frac{N(s)}{D'(s)}$
-1	$-\frac{1}{6}$
3	$\frac{7}{2}$
2	$-\frac{4}{3}$

- $\mathcal{L}^{-1}\left\{\frac{2s^2-4}{s^3-4s^2+s+6}\right\} = -\frac{1}{6}e^{-t} + \frac{7}{2}e^{3t} - \frac{4}{3}e^{2t}$

Example

Evaluate $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^3-s^2+s-1}\right\}$

- $D(s) = 0$
- $s_{1,2,3} = 1, i, -i$
- $N(s) = 3s + 1$
- $D'(s) = 3s^2 - 2s + 1$

s	$\frac{N(s)}{D'(s)}$
1	2
i	$-1 - \frac{1}{2}i$

- $\alpha = -1$
- $\beta = -\frac{1}{2}$
- $a = 0$
- $\omega = 1$
- $\mathcal{L}^{-1}\left\{\frac{3s+1}{s^3-s^2+s-1}\right\} = 2e^t - 2\cos t + \sin t$
 - $= e^{at}(2\alpha \cos \omega t - 2\beta \sin \omega t)$

Example

Evaluate

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

- $= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \frac{1}{s^2+1}\right\}$
- $F(s) = G(s) = \frac{1}{s^2+1}$
- $f(t) = g(t) = \sin t$
- $\int_0^t f(\beta)g(t-\beta)d\beta = \int_0^t \sin \beta \sin(t-\beta)d\beta$
 - $= \int_0^t \sin \beta (\sin t \cos \beta - \cos t \sin \beta)d\beta$
 - $= \int_0^t (\sin \beta \sin t \cos \beta - \cos t \sin^2 \beta)d\beta$
 - $= \sin t I_1 - \cos t I_2$
 - $I_1 = \int_0^t \sin \beta \cos \beta d\beta$
 - $u = \sin \beta$
 - $du = \cos \beta d\beta$
 - $\int u du = \frac{u^2}{2} = \frac{\sin^2 \beta}{2}$
 - $I_2 = \int_0^t \sin^2 \beta d\beta$
 - $\sin^2 \beta = \frac{1}{2}(1 - \cos 2\beta)$
 - $I_2 = \frac{1}{2}\left(\beta - \frac{1}{2}\sin 2\beta\right)$
 - $= \left[\frac{\sin t \sin^2 \beta}{2} - \frac{\cos t \left(\beta - \frac{1}{2}\sin 2\beta\right)}{2}\right]_{\beta=0}^{\beta=t}$
 - $= \left(\frac{\sin t \sin^2 t}{2} - \frac{\cos t \left(t - \frac{1}{2}\sin 2t\right)}{2}\right) - 0$
 - $= \frac{1}{2}(\sin^3 t - \cos t(t - \sin t \cos t))$
 - $= \frac{1}{2}(\sin^3 t - t \cos t + \sin t \cos^2 t)$
 - $= \frac{1}{2}(\sin t(\sin^2 t + \cos^2 t) - t \cos t)$
 - $= \frac{1}{2}(\sin t - t \cos t)$
 - Alternatively, if there are choices...

Inverse Laplace Calculator Approximation

- $F(s)$ will be given...
- Calculate $F(8)$ in calculator.
- In the choices, there will be $f(t)$
- Calculate $\int_0^9 e^{-8t} f(t) dt$
- Answer will be what matched...
 - $F(8) \approx \int_0^9 e^{-8t} f(t) dt$

E) Power Series

Taylor's Series

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f^{(3)}(a)\frac{(x-a)^3}{3!} + \dots$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Maclaurin's Series

$$a = 0$$

$$f(x) = f(a) + f'(a)x + f''(a)\frac{x^2}{2!} + f^{(3)}(a)\frac{x^3}{3!} + \dots$$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k$$

Common Maclaurin's Series

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\cos x = \frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\sin x = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\ln(1+x) = \frac{x^1}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\arctan x = \frac{x^1}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\frac{1}{1+x} = x^0 - x^1 + x^2 - x^3 + \dots$$

Example

What is the coefficient of x^3 in the Taylor's Series polynomial for $\frac{1}{1+2x}$ about $x = 0$

- $a = 0$
- $f(x) = (2x+1)^{-1}$
- $f'(x) = -2(2x+1)^{-2}$
- $f''(x) = 4 \cdot 2(2x+1)^{-3}$
- $f^{(3)}(x) = -24 \cdot 2(2x+1)^{-4} = -48(2x+1)^{-4}$
- $f^{(3)}(a) = -48$
- $C = \frac{f^{(3)}(a)}{3!}$
- $C = \frac{-48}{6}$
- $C = -8$

Example

From the powers of x series for e^x , $\cos x$, and $\sin x$, which of the following relations can be derived?

- (a) $\cos^2 x + \sin^2 x = 1$
- (b) $1 - \cos^2 x = \sin^2 x$
- (c) $\cos x + j \sin x = e^{jx}$
- (d) $\sin x = \frac{e^{jx} - e^{-jx}}{j2}$
- The square of infinite series is not feasible. Therefore, scratch choices "a" and "c".
- Trying c...
- $e^{jx} = \frac{(jx)^0}{0!} + \frac{(jx)^1}{1!} + \frac{(jx)^2}{2!} + \frac{(jx)^3}{3!} + \frac{(jx)^4}{4!} + \frac{(jx)^5}{5!} \dots$
- $e^{jx} = \frac{x^0}{0!} + j\frac{x}{1!} - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} + \dots$
- $e^{jx} = \left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) + j \left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)$
- (c) $e^{jx} = \cos x + j \sin x$

Example

Which of the following is equal to the derivative of $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$?

- (a) $\sin x$
- (b) $\cos x$
- (c) $\arctan x$
- (d) e^x
- $f'(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
- $f'(x) = \cos x$ (b)

Example

Find the equivalent powers of x series for $\arctan x^2$.

- $\arctan x = \frac{x^1}{1} - \frac{x^3}{3} + \frac{x^5}{5} - \dots$
- $\arctan x^2 = \frac{x^2}{1} - \frac{x^6}{3} + \frac{x^{10}}{5} - \dots$

Example

Find the first 3 terms in the power series of $\frac{e^x}{1-x}$.

- $f(x) = \frac{e^x}{1-x} \rightarrow f(0) = 1$
- $f'(x) = \frac{e^x(1-x-(-1))}{(1-x)^2} = \frac{e^x(2-x)}{(1-x)^2} \rightarrow f'(0) = 2$
- $f''(0) = \frac{d}{dx} \left(\frac{e^x(2-x)}{(1-x)^2} \right) \Big|_{x=0} = 5$
- $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} x^k$
- $f(x) = \frac{1x^0}{0!} + \frac{2x^1}{1!} + \frac{5x^2}{2!} + \dots$
- $f(x) = 1 + 2x + 2.5x^2$
- Alternatively, if there are choices...
 - Solve for $f(x_0)$ using the given function $f(x)$ and the power series $f(x)$ in the choices and see which will match.
 - x_0 can be chosen randomly.

F) The Fourier Series

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \right)$$

$$a_0 = \frac{1}{T} \int_T f(x) dx$$

$$a_n = \frac{1}{T} \int_{-T}^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

$$b_n = \frac{1}{T} \int_{-T}^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

- Sum of cosine and sine functions used to represent a function $f(x)$
- Similar to a Taylor series
- $\frac{a_0}{2}$ is the average value of the function over its period.

- 3 Conditions for $f(x)$ intended for Fourier Series with a period of $2T$:**

- $f(x)$ must be defined in the interval:
 $c < x < c + 2T$
- $f(x)$ and $f'(x)$ are sectionally continuous in the interval:
 $c < x < c + 2T$
- $f(x + 2T) = f(x)$

Fourier Series for a Period of 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx$$

Half Range Fourier Cosine and Sine Series

Half Range Fourier Cosine Series

if $f(x)$ is even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{T}\right) \right)$$

$$a_0 = \frac{2}{T} \int_0^T f(x) dx$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

$$b_n = 0$$

Half Range Fourier Sine Series

if $f(x)$ is odd

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin\left(\frac{n\pi x}{T}\right) \right)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

- Integral of:
 - even $\times$ even $\neq 0$
 - odd $\times$ odd $\neq 0$
 - even $\times$ odd $= 0$
 - odd $\times$ even $= 0$

Example

Find the Fourier series for $f(x) = x^2$ over the interval $-\pi$ to π

- $f(x)$ is even $\therefore$ use half range fourier cosine series.
- $T = \pi$
- $a_0 = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{3\pi} \pi^3 = \frac{2\pi^2}{3}$
- $a_n = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx ; n = 1, 2, 3, \dots$
 - Substitute na sa calcu...
- $a_1 = -4$
- $a_2 = 1$
- $a_3 = -\frac{4}{9}$
- $b_n = 0$
- $f(x) = \frac{\pi^2}{3} - 4 \cos x + \cos 2x - \frac{4}{9} \cos 3x + \dots$
 - First term is $\frac{a_0}{2}$

Example

Find b_3 for the Fourier sine series of $f(x) = x^2$ over the interval $0 < x < 1$

- $b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{n\pi x}{T}\right) dx$
- $b_3 = \frac{2}{1} \int_0^1 x^2 \sin\left(\frac{3\pi x}{1}\right) dx$
- $b_3 \approx 0.2027$
 - From calculator

G) Matrices and Determinants

Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- A rectangular array of numbers called elements which are arranged in rows (m) and in columns (n)
- Its dimension is $m \times n$
- Ex:

$$\circ [1 \quad 2 \quad 3] \text{ is } 1 \times 3$$

$$\circ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \text{ is } 2 \times 2$$

• Square Matrix

- Matrix with the same number of rows and columns
- $m = n$

• Identity Matrix

- Square matrix comprising of ones in the main diagonal and zeroes for the rest.
- Ex:

$$\blacksquare \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

• Equal Matrices

- Same dimensions
- Same elements on respective positions
 - If $A = B$, the $a_{mn} = b_{mn}$

• Transposed Matrix, A^T

- The resulting matrix when elements in rows were made in column by using elements in the main diagonal as pivot elements.
- When transposing, main diagonal is unchanged.
- When transposing, determinant is unchanged.

• Determinant of a Matrix, $\det A$ or $|A|$

- A constant associated with a given square matrix.

• Minor of an Element, M_{mn}

- The resulting determinant after removing the elements on its row and column

• Cofactor of an Element, C_{mn}

- Signed minor for the same element
- Sign is $(-1)^{m+n}$

- Sign can also be taken from the matrix of alternating signs

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

- **Upper Triangular Matrix**

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix}$$

- The elements below the main diagonal are all zeroes.
- Its determinant is the product of the main diagonal.
 - If minor-cofactor method is used, the same result will arise.

- **Eigenvalues, λ**

$$|A - \lambda I| = 0$$

- The value of a variable determined by subtracting the variable from the elements of the main diagonal of a square matrix and then equating the determinant to zero

- **Eigenvector, v**

$$Av = \lambda v$$

$$(A - \lambda I)v = Z$$

- The column matrix that when multiplied to a given matrix with the eigen value subtracted from the main diagonal equals or results to a zero matrix (Z).
- The column matrix that if multiplied to the scalar eigenvalue or the matrix yields the same matrix.

Example

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then the transposed matrix is:

- $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

Example

Find the determinant of the matrix A in the previous example.

- $|A| = 1 \times 4 - 3 \times 2 = 4 - 6$
- $|A| = -2$

Example

Given is matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 4 \\ -1 & 0 & 2 \end{pmatrix}$. Find the minor of the element a_{22} .

- $M_{22} = \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix}$
- $M_{22} = 1 \times 2 - 3 \times (-1) = 2 + 3$
- $M_{22} = 5$

Example

Given is matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \\ -1 & 0 & 2 \end{bmatrix}$, find the cofactor of a_{23} .

- $C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} = -(0 - (-2))$
- $C_{23} = -2$

Example

Find the eigenvalues for the matrix $\begin{pmatrix} 0 & 5 \\ 2 & 3 \end{pmatrix}$.

- $|A - \lambda I| = 0$
- $\left| \begin{pmatrix} 0 & 5 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$
- $\begin{vmatrix} -\lambda & 5 \\ 2 & 3 - \lambda \end{vmatrix} = 0$
- $(-\lambda)(3 - \lambda) - (5)(2) = 0$
- $\lambda^2 - 3\lambda - 10 = 0$
- $(\lambda - 5)(\lambda + 2) = 0$
- $\lambda_{1,2} = 5, -2$

Example

Given is matrix $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$, find the eigen vectors.

- $\begin{vmatrix} -\lambda & 1 \\ -2 & 3 - \lambda \end{vmatrix} = 0$
- $\lambda^2 - 3\lambda + 2 = 0$
- $(\lambda - 2)(\lambda - 1) = 0$
- $\lambda_{1,2} = 2, 1$
- $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$
- $y = 2x$
- $-2x + 3y = 2y \rightarrow y = 2x$
- Non-trivial solution.
- $v_1 = \begin{pmatrix} x \\ 2x \end{pmatrix}$

- $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1 \begin{pmatrix} x \\ y \end{pmatrix}$
- $y = x$
- $-2x + 3y = y \rightarrow y = x$
- Non-trivial solution.
- $v_2 = \begin{pmatrix} x \\ x \end{pmatrix}$
- $v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Expansion of Determinants

- Solutions:
 - Minor-Cofactor Method
 - Laplace Expansion Method

Example

Expand the 4th order (4×4) matrix determinant.

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 1 & 6 & 4 & -2 \end{bmatrix}$$



$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

- $\begin{vmatrix} 3 & 2 & 1 \\ -1 & 2 & 3 \\ 6 & 4 & -2 \end{vmatrix} - 4 \begin{vmatrix} 2 & 3 & 4 \\ -1 & 2 & 3 \\ 6 & 4 & -2 \end{vmatrix} - \begin{vmatrix} 2 & 3 & 4 \\ 3 & 2 & 1 \\ -1 & 2 & 3 \end{vmatrix}$
- Store the matrices in matrix A, B, and C in calculator.
- $Det(A) - 4Det(B) - Det(C)$
- **= 150**

Example

Expand the 5th order determinant.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 3 & 4 & 5 \\ 0 & 0 & 3 & 4 & 5 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- Just multiply the main diagonal
- $= (1)(2)(3)(-4)(1)$
- **= -24**

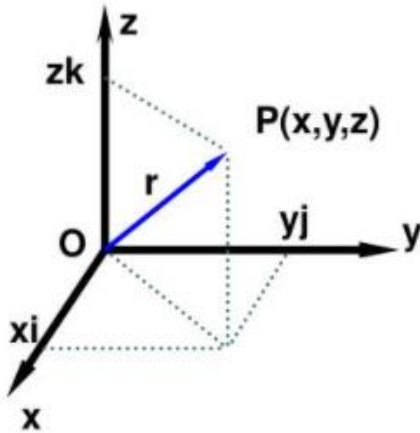
VECTOR ANALYSIS

vector analysis

1) Definition of Terms

Position Vector, $\vec{r}$

- Vector that starts from origin and ends at a point



Forms

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \quad \text{Algebraic Form}$$

$$\vec{r} = [x, y, z] \quad \text{Analytical / Matrix Form}$$

$$\vec{r} = \langle x, y, z \rangle \quad \text{Analytical / Matrix Form}$$

Magnitude

(length of given vector)

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Unit Vector

(vector with magnitude of one in the direction of a given vector)

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z^2}}$$

Example

Given is a vector that starts from the point $(3, -2, 1)$ and terminates at the point $(2, 4, -2)$. Find:

- The equivalent position vector
 - The magnitude
 - The unit vector
- $\vec{r} = [2, 4, -2] - [3, -2, 1]$
 - End point minus starting point
 - (a) $\vec{r} = [-1, 6, -3] = -\hat{i} + 6\hat{j} - 3\hat{k}$
 - $|\vec{r}| = \sqrt{1^2 + 6^2 + 3^2}$
 - (b) $|\vec{r}| = \sqrt{46}$
 - $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{-\hat{i} + 6\hat{j} - 3\hat{k}}{\sqrt{46}}$
 - (c) $\hat{r} = -\frac{1}{\sqrt{46}}\hat{i} + \frac{6}{\sqrt{46}}\hat{j} - \frac{3}{\sqrt{46}}\hat{k}$

Note: All points and vectors in space can be represented by a position vector.

II) Vectors

Operations on Vectors

$$\vec{A} = x_a \hat{i} + y_a \hat{j} + z_a \hat{k}$$

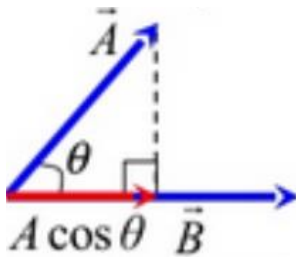
$$\vec{B} = x_b \hat{i} + y_b \hat{j} + z_b \hat{k}$$

$$\vec{C} = x_c \hat{i} + y_c \hat{j} + z_c \hat{k}$$

Addition / Subtraction

$$\vec{A} \pm \vec{B} = (x_a \pm x_b)\hat{i} + (y_a \pm y_b)\hat{j} + (z_a \pm z_b)\hat{k}$$

Dot (Scalar) Product



$$\vec{A} \cdot \vec{B} = x_a x_b + y_a y_b + z_a z_b$$

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$$

Acute Angle Between Vectors

$$\theta = \arccos \left| \frac{\vec{A} \cdot \vec{B}}{|\vec{A}| |\vec{B}|} \right|$$

Perpendicular Vectors

$$\vec{A} \cdot \vec{B} = 0 \text{ if } \vec{A} \perp \vec{B}$$

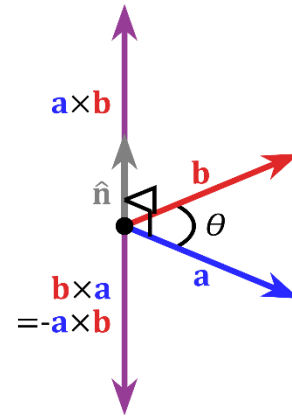
Work Done in Moving a Body

$$W = \vec{F} \cdot \vec{d}$$

F = force vector

d = displacement vector

Cross (Vector) Product



$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_a & y_a & z_a \\ x_b & y_b & z_b \end{vmatrix}$$

$$\vec{A} \times \vec{B} = (|\vec{A}| |\vec{B}| \sin \theta) \hat{n}$$

Note:

Cross product is a vector perpendicular to both vectors multiplied.

Unit Normal Vector to Plane of Two Vectors

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{\vec{A} \times \vec{B}}{|\vec{A}| |\vec{B}| \sin \theta}$$

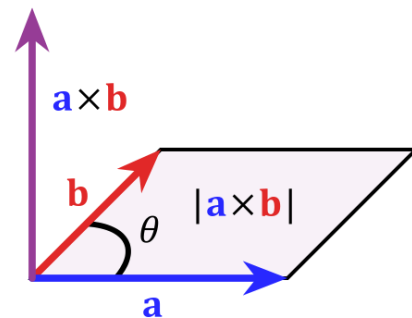
Cross Product of Unit Vectors

$$\hat{i} \times \hat{j} = \hat{k}$$

$$\hat{j} \times \hat{k} = \hat{i}$$

$$\hat{k} \times \hat{i} = \hat{j}$$

Area of Parallelogram / Triangle with 2 Sides Represented by $\vec{A}$ and $\vec{B}$



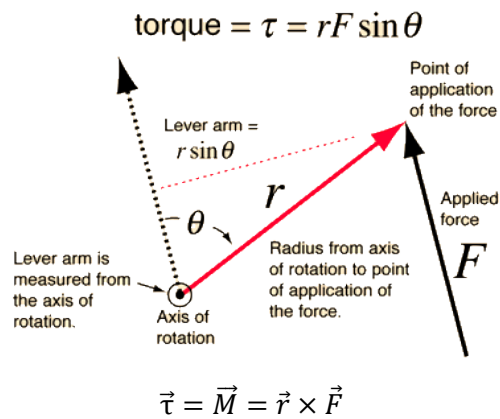
$$A_P = |\vec{A} \times \vec{B}|$$

Parallelogram

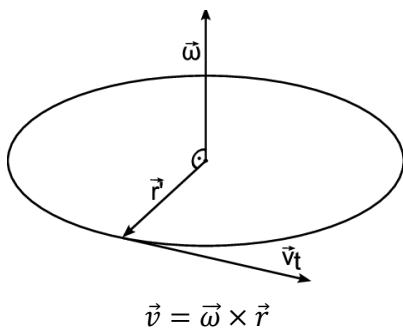
$$A_T = \frac{1}{2} |\vec{A} \times \vec{B}|$$

Triangle

Torque or Moment of a Force



Tangential Velocity

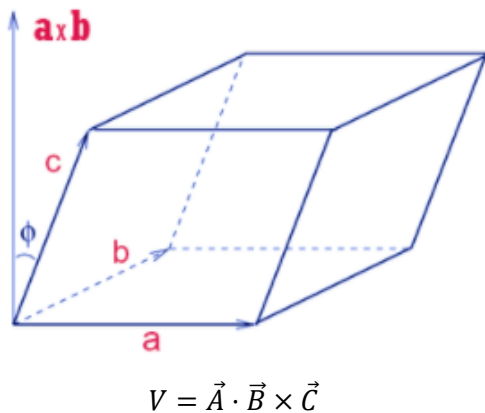


Scalar Triple Product

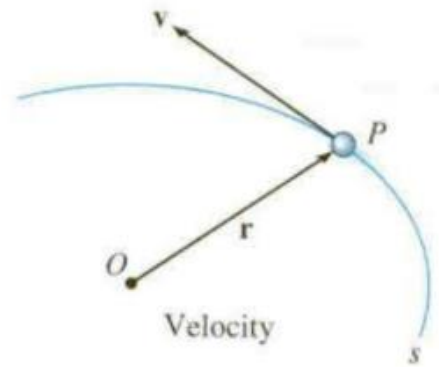
$$\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{vmatrix}$$

$$\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \cdot \vec{A}$$

Volume of Parallelepiped with 3 Edges represented by vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$



Acceleration and Velocity in a Space Curve



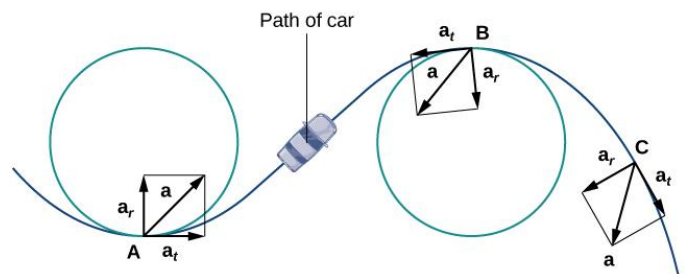
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{v} = \vec{r}' = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \vec{v}' = \vec{r}'' = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Note: Velocity $\vec{v}$ is tangential velocity

Tangential and Normal Acceleration



$$a_t = \vec{a} \cdot \hat{v} = \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|}$$

$$a_n = \sqrt{a^2 - a_t^2}$$

$$\vec{a}_n + \vec{a}_t = \vec{a}$$

Del (Differential) Operator

Has no specific use in itself. Used for gradient of a scalar, and divergence and curl of a vector

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

Gradient

Vector quantity which is outward and normal to the surface defined by scalar field $\phi = f(x, y, z)$

$$\text{grad} \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k}$$

Maximum Slope on Surface of ϕ

$$\phi'_{\max} = |\nabla \phi|$$

Directional Derivative

On the direction of $\vec{u}$

$$D_u f(x, y, z) = \nabla \phi \cdot \hat{u} = \nabla \phi \cdot \frac{\vec{u}}{|\vec{u}|}$$

Divergence and Curl

$$\vec{V} = f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$$

$$\text{div} \vec{V} = \nabla \cdot \vec{V} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

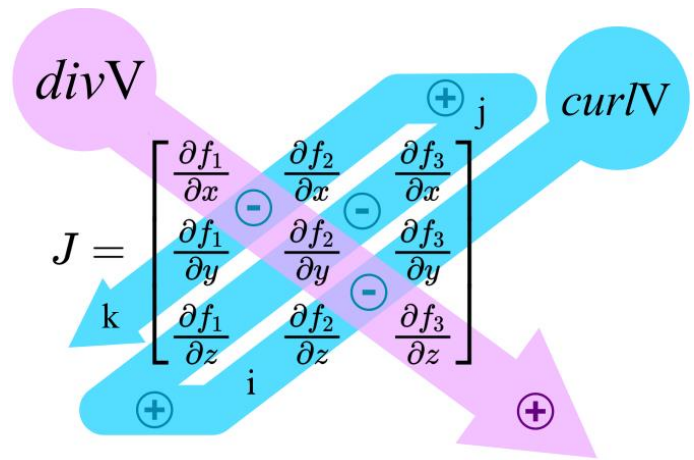
$$\text{curl} \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{curl} \vec{V} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \right) \hat{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z} \right) \hat{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) \hat{k}$$

Note:

- Divergence of a vector is a scalar (field).
- Curl of a vector is a vector (field).

Jacobian Matrix



Example

Given are 3 forces $A = 3\hat{i} - 2\hat{j} + \hat{k}$, $B = 2\hat{i} + 4\hat{j} - 2\hat{k}$, and $C = -\hat{i} + 3\hat{j} + 2\hat{k}$. Find:

- The resultant force vector
- The magnitude of the resultant force

- $\vec{R} = A + B + C$
- (a) $\vec{R} = 4\hat{i} + 5\hat{j} + \hat{k}$
- $|\vec{R}| = \sqrt{4^2 + 5^2 + 1^2}$
- (b) $|\vec{R}| = \sqrt{42}$

Example

If $A = 3\hat{i} - 2\hat{j} + \hat{k}$ and $B = 2\hat{i} + 4\hat{j} - 2\hat{k}$, find:

- $A \cdot B$
- The acute angle between A and B

- $A \cdot B = 3 \cdot 2 + (-2) \cdot 4 + 1 \cdot (-2)$
- (a) $A \cdot B = -4$
- $\theta = \arccos \left| \frac{A \cdot B}{|A||B|} \right|$
- $\theta = \arccos \left| \frac{-4}{\sqrt{3^2 + 2^2 + 1^2} \sqrt{2^2 + 4^2 + 2^2}} \right|$
- (b) $\theta = 77.40^\circ$

Example

Find the values of x so that $A = xi - 5j + 2k$ and $B = xi + xj + 3k$ are perpendicular vectors.

- $A \cdot B = 0$
- $x^2 - 5x + 6 = 0$
- $(x - 3)(x - 2) = 0$
- $x = 3, 2$

Example

Find the work done in moving a particle from $(3, -2, 1)$ to $(2, 4, -2)$ by the force $F = -i + 3j + 2k$. Unit of distance is in meter (m) and unit of force is in Newton (N).

- $W = \vec{F} \cdot \vec{d}$
- $\vec{d} = (2, 4, -2) - (3, -2, 1)$
- $\vec{d} = (-1, 6, -3)$
- $W = (-1, 3, 2) \cdot (-1, 6, -3)$
- $W = 1 + 18 - 6$
- $W = 13 \text{ N} \cdot \text{m}$

Example

If $A = 3i - 2j + k$ and $B = 2i + 4j - 2k$. Find:

- Unit vector orthogonal to both A and B
- The area of the triangle with 2 sides represented by A and B

- $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 1 \\ 2 & 4 & -2 \end{vmatrix}$
- $\vec{A} \times \vec{B} = (4 - 4)\hat{i} - (-6 - 2)\hat{j} + (12 - (-4))\hat{k}$
- $\vec{A} \times \vec{B} = 8\hat{j} + 16\hat{k}$
- $\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{8\hat{j} + 16\hat{k}}{\sqrt{8^2 + 16^2}}$
- (a) $\hat{n} = \frac{1}{\sqrt{5}}\hat{j} + \frac{2}{\sqrt{5}}\hat{k}$
- $A_T = \frac{1}{2}|\vec{A} \times \vec{B}|$
- $A_T = \frac{1}{2}\sqrt{8^2 + 16^2}$
- $A_T = \frac{1}{2}(8\sqrt{5})$
- (b) $A_T = 4\sqrt{5}$

Example

Find the area of that part of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ in the first octant.

- We have three points (the intercepts)
 - $(a, 0, 0)$
 - $(0, b, 0)$
 - $(0, 0, c)$
- We can get two vectors with the same starting point
 - $\vec{A} = (0, b, 0) - (a, 0, 0) = (-a, b, 0)$
 - $\vec{B} = (0, 0, c) - (a, 0, 0) = (-a, 0, c)$
- We can get the cross product.
 - $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix}$
 - $\vec{A} \times \vec{B} = bc\hat{i} + ac\hat{j} + ab\hat{k}$
- We can get the area
 - $A_T = \frac{1}{2}|\vec{A} \times \vec{B}|$
 - $A_T = \frac{1}{2}\sqrt{(bc)^2 + (ac)^2 + (ab)^2}$

Example

Find the volume of the parallelepiped with 3 edges represented by vectors

$$\vec{A} = 3i - 2j + k$$

$$\vec{B} = 2i + 4j - 2k$$

$$\vec{C} = -i + 3j + 2k$$

- $V = \vec{A} \cdot \vec{B} \times \vec{C}$
- $V = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 4 & -2 \\ -1 & 3 & 2 \end{vmatrix}$
- $V = 56 \text{ units}^3$

Example

A force given by $F = 3i + 2j - 4k$ is applied at the point $(1, -1, 2)$. Find the magnitude of the moment of F about the point $(2, -1, 3)$.

- $\vec{M} = \vec{r} \times \vec{F}$
- $\vec{r} = (1, -1, 2) - (2, -1, 3) = (-1, 0, -1)$
 - End point minus starting point
 - End point is the point where force is applied

- Starting point is center of the moment

- $\vec{M} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$
- $\vec{M} = 2\hat{i} - 7\hat{j} - 2\hat{k}$
- $|\vec{M}| = \sqrt{2^2 + 7^2 + 2^2} = \sqrt{57}$
- $|\vec{M}| = 7.55$

Example

The angular velocity of a rotating rigid body about an axis of revolution is given by $\omega = 4\hat{i} + \hat{j} - 2\hat{k}$. Find the linear velocity of a point on the body whose position vector relative to a point on the axis of rotation is $r = 2\hat{i} - 3\hat{j} + \hat{k}$.

- $\vec{v} = \vec{\omega} \times \vec{r}$
- $\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 1 & -2 \\ 2 & -3 & 1 \end{vmatrix}$
- $\vec{v} = -5\hat{i} - 8\hat{j} - 14\hat{k}$

Extra: Find the magnitude of the linear velocity.

- $|\vec{v}| = \sqrt{5^2 + 8^2 + 14^2}$
- $|\vec{v}| = 16.88$

Example

Find the magnitude of the tangential velocity and total acceleration of a particle moving on the curve defined by $x = 2 \sin 3t$, $y = 2 \cos 3t$, $z = 8t$ at anytime $t > 0$.

- $\vec{r} = \langle x \ y \ z \rangle$
- $\vec{r} = \langle 2 \sin 3t \ 2 \cos 3t \ 8t \rangle$
- $\vec{v} = \vec{r}'$
- $\vec{v} = \langle 6 \cos 3t \ -6 \sin 3t \ 8 \rangle$
- $\vec{a} = \vec{v}'$
- $\vec{a} = \langle -18 \sin 3t \ -18 \cos 3t \ 0 \rangle$
- $|\vec{v}| = \sqrt{36 \cos^2 3t + 36 \sin^2 3t + 64}$
- $|\vec{v}| = \sqrt{36 + 64}$
- $|\vec{v}| = 10$
- $|\vec{a}| = \sqrt{18^2 \sin^2 3t + 18^2 \cos^2 3t}$
- $|\vec{a}| = 18$

Example

A particle moves along the curve $r = (t^3 - 4t)\hat{i} + (t^2 + 4t)\hat{j} + (8t^2 - 3t^3)\hat{k}$, where t is the time. Find the magnitude of the tangential and normal components of the acceleration when $t = 2$.

- $\vec{r} = \langle t^3 - 4t \ t^2 + 4t \ 8t^2 - 3t^3 \rangle$
- $\vec{v} = \vec{r}' = \langle 3t^2 - 4 \ 2t + 4 \ 16t - 9t^2 \rangle$
- $\vec{a} = \vec{v}' = \langle 6t \ 2 \ 16 - 18t \rangle$
- At $t = 2$
- $\vec{v} = \langle 8 \ 8 \ -4 \rangle$
- $\vec{a} = \langle 12 \ 2 \ -20 \rangle$
- $a_t = \vec{a} \cdot \hat{v} = \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|} = \frac{12 \cdot 8 + 2 \cdot 8 + (-20) \cdot (-4)}{\sqrt{8^2 + 8^2 + 4^2}}$
- $a_t = 16$
- $a = |\vec{a}| = \sqrt{12^2 + 2^2 + 20^2} = 23.41$
- $a_n = \sqrt{a^2 - a_t^2} = \sqrt{548 - 16^2}$
- $a_n = 2\sqrt{73} = 17.09$

Example

Find the magnitude of the gradient of $f(x, y, z) = x^2 + y^2 + z^2$ at the point $(3, 2, 1)$.

- $\nabla f = \langle 2x \ 2y \ 2z \rangle$
- At point $(3, 2, 1) \dots$
- $\nabla f = \langle 6 \ 4 \ 2 \rangle$
- $|\nabla f| = \sqrt{6^2 + 4^2 + 2^2}$
- $|\nabla f| = 7.48$

Example

The electric potential at any point (x, y, z) is given by $V = x^2 + 4y^2 + 9z^2$. Find the rate of change of potential at point $(2, -1, 3)$ towards the origin.

- We can get the direction u .
- End point is origin $(0, 0, 0)$
- Starting point is $(2, -1, 3)$
- $\vec{u} = (0, 0, 0) - (2, -1, 3) = (-2, 1, -3)$
- $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{(-2, 1, -3)}{\sqrt{2^2 + 1^2 + 3^2}} = \left(-\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, -\frac{3}{\sqrt{14}}\right)$
- $\nabla V = (2x, 8y, 18z) = (4, -8, 54)$
- $D_u V(2, -1, 3) = \nabla V \cdot \hat{u} = \frac{-8 - 8 - 162}{\sqrt{14}}$
- $D_u V(2, -1, 3) = \frac{-178}{\sqrt{14}}$

Example

The electrical potential at any point (x, y, z) is given by $V = x^2 + 4y^2 + 9z^2$. Find the maximum rate of change of the potential at the point $(2, -1, 3)$.

- $V'_{max} = |\nabla V|$
- $\nabla V = \langle 2x \quad 8y \quad 18z \rangle = \langle 4 \quad -8 \quad 54 \rangle$
- $V'_{max} = \sqrt{4^2 + 8^2 + 54^2}$
- $V'_{max} = \mathbf{54.74}$

Example

Find the divergence and magnitude of curl on the vector field:

$$V = (x^2y)i + (-xy)j + (xyz)k$$

At the point $(3, 2, 1)$

- Constructing Jacobian Matrix...
- $J = \begin{bmatrix} 2xy & -y & yz \\ x^2 & -x & xz \\ 0 & 0 & xy \end{bmatrix} = \begin{bmatrix} 12 & -2 & 2 \\ 9 & -3 & 3 \\ 0 & 0 & 6 \end{bmatrix}$
- $\text{div } \vec{V} = 12 - 3 + 6$
- $\mathbf{\text{div } \vec{V} = 15}$
- $\text{curl } \vec{V} = (3 - 0)\hat{i} + (0 - 2)\hat{j} + (-2 - 9)\hat{k}$
- $\text{curl } \vec{V} = 3\hat{i} - 2\hat{j} - 11\hat{k}$
- $|\text{curl } \vec{V}| = \sqrt{3^2 + 2^2 + 11^2}$
- $\mathbf{|\text{curl } \vec{V}| = 11.58}$

STATISTICS & PROBABILITY

statistics & probability

1) The Fundamental Principle of Counting

Fundamental Principle of Counting

If:

1st event can happen in m ways

2nd event can happen in n ways

They happen together OR in succession

Then:

They will happen in $m \times n$ ways.

For more than two events: $m \times n \times \dots$

Example

A building has 6 doors. In how many ways can a man enter and leave the building:

(a) By any door.

(b) By a different door.

- For a, a man can enter any of the 6 doors and can leave at any of the 6 doors. Thus, the man can enter in 6 ways and leave in 6 ways. $W = 6 \times 6$
- (a) $W = 36$ ways**
- For b, a man can enter any of the 6 doors, and then can only leave at any of the 5 doors except the one he entered. Thus, the man can enter in 6 ways and leave in 5 ways. $W = 6 \times 5 = 30$
- (b) $W = 30$ ways**

Example

The license plate of a car contains 3 letters and a 3-digit number. How many license plates can be made without repetition?

- There are 26 letters in the alphabet (A to Z) and 10 numerals (0 to 9).
- The number of ways each letter and numerical digit can be chosen is:

L1	L2	L3	N1 (100's)	N2 (10's)	N3 (1's)
26	25	24	9	9	8

- The succeeding letter are less one way because the previous letter used cannot be used.
- The first number (hundreds place) should be non-zero in order for it to be a "3-digit number". Thus, only 9 digits can be used.
- The second number can now use the number "0" but cannot use the previous numeral used. Thus, the second number (tens place) can use 9 digits also.
- The third number can use 8 digits since it cannot use the previous numeral.
- $W = 26 \times 25 \times 24 \times 9 \times 9 \times 8$
- $W = 10\ 108\ 800$ ways**

Example

A code is composed of 2 letters, the first being a vowel and 3-digits. In how many ways can it be made without repetition?

L1	L2	N1	N2	N3
5	25	10	9	8
5 vowel letters	26 less 1	10 numerals ("0" can be used)	10 less 1	9 less 1

- $W = 5 \times 25 \times 10 \times 9 \times 8$
- $W = 90\ 000$ ways**

Example

In how many ways can 4 boys and 5 girls be seated in a bench alternately?

- If you start with boys, there will be two girls at the end of the bench.
- A girl should be seated first.

G ₁	B ₁	G ₂	B ₂	G ₃	B ₃	G ₄	B ₄	G ₅
5		4		3		2		1
	4		3		2		1	

- We subtract one for the next boy or girl since a boy or a girl cannot be seated at the same seat at the same time.
- $W = 5! \times 4!$
- $W = 2880$ ways**

Example

How many 4-digit even numbers can be formed from 0 to 9 if each digit is to be used only once in each number?

- For it to be even, ones place should be 0, 2, 4, 6, or 8.
- For it to be a 4-digit number, the thousands place should be non-zero.
- Two cases should be considered.
 - Ones place = 0 & Thousands place $\neq 0$
 - Ones place $\neq 0$ & Thousands place $\neq 0$

	1000's	100's	10's	1's
Case 1	9	8	7	1
	No "0"			"0"
Case 2	8	8	7	4
	No "0" and no numeral used in 1's (10 - 2)	With "0" but without numeral used in 1000's (8+1-1)	"2,4,6,8"	

- $W = W_1 + W_2 = 9 \times 8 \times 7 \times 1 + 8 \times 8 \times 7 \times 4$
- $W = 2\,296$ ways**
- (Alternative method)
- $W = W_T - W'$
 - All 4-digit numbers less all odd numbers equals all even numbers.

	1000's	100's	10's	1's
All Nos.	9	9	8	7
	No "0" (10-1)	With "0" but without numeral used in 1000's (10+1-1)	(9-1)	(8-1)
All Odd	8	8	7	5
	No "0" and no numeral used in 1's (10 - 2)	With "0" but without numeral used in 1000's (8+1-1)	(8-1)	"1,3,5,7,9"

- $W = W_T - W' = 9 \cdot 9 \cdot 8 \cdot 7 - 8 \cdot 8 \cdot 7 \cdot 5$
- $W = 2\,296$ ways**

II) The Permutation

Permutation

- It deals with the arrangement of elements with order.
- Denoted as
 - $P(n, r)$
 - nPr
 - P_r^n
 - $P_{n,k}$
- Means
 - n permute r .
- Permutation of n different elements **taken r at a time**
 - $nPr = \frac{n!}{(n-r)!}$
 - $nPr = n \cdot (n-1) \cdot (n-2) \cdots r$ factors
- Permutation of n different elements in a row or in line **taken all (n) at a time**
 - $P = nPn = n!$**
- Permutation of n elements with same elements and taken all at a time
 - $P = \frac{n!}{(r_1!)(r_2!)(r_3!) \cdots}$
 - r 's are multiplicity of similar elements

Example

In how many ways can 5 starting positions on a basketball team be filled from 8 men who can play any of the positions?

- $8P5 = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$
 - 5 factors
- $8P5 = 6720$ ways**

Example

In how many ways can 6 people be lined up on a bus?

- $6P6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
 - 6 factors
- $6P6 = 720$ ways**

Example

In how many ways can 6 people be seated in a round table.

- With one seat as a reference point, 5 people can be picked to sit at the remaining five seats.
 - This is because in a given permutation, if the seating arrangement is rotated, it is counted as one permutation as it is still the same order that the people is seated.
- $5P5 = 5!$
- **$5P5 = 120$ ways**
- Formula for number of ways that n people can be seated in a round table
 - $P = (n - 1)!$

Example

How many distinct permutations can be made from the letters of the word infinity?

- The set of letters to be used are $\{i, n, f, i, n, i, t, y\}$ having 8 elements.
 - $n = 8$
- Element “i” has a multiplicity of 3.
 - $r_1 = 3$
- Element “n” has a multiplicity of 2.
 - $r_2 = 2$
- $P = \frac{n!}{(r_1!)(r_2!)}$
- $P = \frac{8!}{3!2!}$
- **$P = 3\ 360$ ways**

Example

In how many ways can 4 boys and 4 girls be seated in:

- (a) A row of 8 chairs alternately?
- (b) A round table alternately
- (a)
 - $P(4 \text{ boys}) = 4!$
 - $P(4 \text{ girls}) = 4!$
 - $P(\text{boy and girl}) = 2!$
 - Boy can go first or girl can go first.
 - $W = 4! \cdot 4! \cdot 2!$
 - **$W = 1\ 152$ ways**

- (b)
 - One should be reference element. Consider one boy as reference element.
 - $P(3 \text{ boys}) = 3!$
 - $P(4 \text{ girls}) = 4!$
 - A girl is always picked first to sit beside the reference element that is a boy. A boy will never be picked first.
 - **$W = 3! \cdot 4! = 144$ ways**

Example

In how many ways can 3 boys and 4 girls be seated in a bench with the boys always together?

- $P(3 \text{ boys}) = 3!$
- $P(1 \text{ grp of three boys and } 4 \text{ girls}) = 5!$
- $W = 5! \cdot 3!$
- **$W = 720$ ways**

Example

Six people are lined up on a bus. In how many ways can the people be arranged if two of them do not want to follow each other?

- $P(2 \text{ people}) = 2!$
- $P(4 \text{ people and } 1 \text{ grp of two people}) = 5!$
- $W' = 2! 5!$
 - Ways that six people can be lined up with two people always following each other.
- $W = W_T - W'$
- $W = 6! - 2! \cdot 5!$
- **$W = 480$ ways**

III) The Combination

Combination

- It deals with the arrangements of different elements without order.
- Denoted as
 - $C(n, r)$
 - nCr
 - C_r^n
 - $C_{n,k}$
- Means
 - n choose r
- Combination of n different elements **taken r at a time**
 - $nCr = \frac{n!}{r!(n-r)!}$
 - $nCr = \frac{n \cdot (n-1) \cdot (n-2) \cdots r \text{ factors}}{r!}$
- Combination of n different elements **taken 1 or 2 or 3 or ... n at a time.**
 - $C = nC1 + nC2 + nC3 + \cdots nCn$
 - Can be calculated in calculator as:
 - $\sum_{x=1}^n (nC_x)$
 - $C = 2^n - 1$
- Combination of n different elements in a row or in line **taken all (n) at a time**
 - $C = nCn = 1$

Example

In how many ways are there to select 3 candidates from 8 equally qualified applicants for the employment opening in a company?

- $8C3 = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1}$
- $8C3 = 56$ ways**

Example

In how many ways can a poster be colored if there are 5 different colors which are available?

- 5 different colors can be chosen 1, 2, 3, 4, or 5 at a time.
- $C = 2^5 - 1$
- $C = 31$ ways**

Example

In how many ways can a majority be formed from 5 judges?

- Majority will be formed when 3, 4, or 5 of the judges is in favor.
- $C = 5C3 + 5C4 + 5C5$
- $C = \sum_{x=3}^5 5C_x$
- $C = 16$ ways**

Example

From a group of 12 players, in how many ways can the first five be selected if one of them is the captain ball.

- Captain ball is given to be in the first five.
- So we need to choose only 4 from 11 players.
 - 12 – captain ball = 11
 - 5 – captain ball = 4
- $11C4 = \frac{11 \cdot 10 \cdot 9 \cdot 8}{4 \cdot 3 \cdot 2 \cdot 1}$
- $11C4 = 330$ ways**

Example

In how many ways can a committee of 5 be selected from 4 lawyers and 5 engineers if 2 are lawyers and the rest are engineers.

- From 4 lawyers, 2 is chosen.
- From 5 engineers, 3 is chosen
- $C = 4C2 \cdot 5C3$
- $C = 60$ ways**

Example

A shipment of 12 TV sets contains 3 defective sets. In how many ways can you purchase 5 of these TV sets and receive at least 2 of the defective sets?

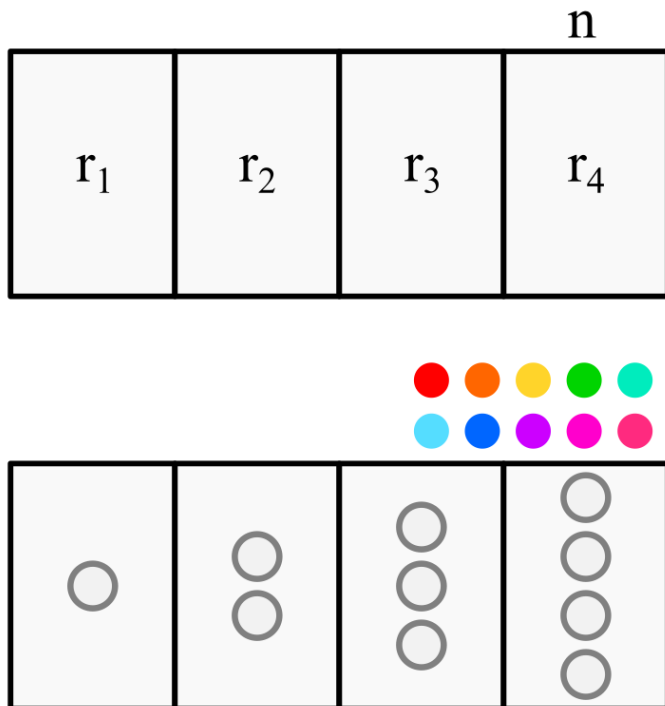
- There are:
 - 3 defective sets.
 - 9 non-defective sets.
- Condition is “purchase 5 TV sets” and “receive at least 2 of the defective sets”. So there can be 2 or 3 defective sets.

	Defective Sets	Non-defective Sets
Case 1: 2 Defective Sets	$3C2$	$9C3$
Case 2: 3 Defective Sets	$3C3$	$9C2$

- In case 1, combination of 2 defective sets is taken from the three defective set. And a combination of 3 non-defective sets is taken from the nine defective sets. There are 5 sets in total.
- In case 2, same concept is used but with 3 defective sets.
- $C = 3C2 \cdot 9C3 + 3C3 \cdot 9C2$
- $C = 288$ ways

IV) The Partition Principle

The Partition Principle



- $W = \frac{n!}{(r_1!)(r_2!)(r_3!) \dots}$
- Understanding the concept
 - The question is how many combination in each partition can be made.
 - In the example above, 10 circles is partitioned into 1 circle, 2 circles, 3 circles, and 4 circles.
 - The first partition has $10C1$ combinations.
 - The second partition has $9C2$ combinations. Only 9 can be used because one is already in first partition.
 - The third partition has $7C3$ combinations. Only 7 can be used because three are on the first and second partition.
 - The fourth partition has $4C4$ combinations. Only 4 can be used

because the six are on the first to third partition.

- Adding these outputs the same number.
- $\frac{10!}{1! \cdot 2! \cdot 3! \cdot 4!} = 10C1 \cdot 10C2 \cdot 10C3 \cdot 10C4$

Example

In how many ways can a hotel accommodate 7 guests if the rooms available are one triple and two double rooms?

- Use
 - $W = 7C3 \cdot 4C2 \cdot 2C2$ OR
 - $W = \frac{7!}{3! \cdot 2! \cdot 2!}$
- $W = 210$ ways

Example

In how many ways can 9 people cross a river riding 3 boats whose maximum capacity is 2, 4, and 5 respectively:

- (a) If all boats must travel?
- (b) If all boats must have passengers?

- List of Cases of Passengers/Capacity
 - (Passengers must add up to 9 people)
 - $2/2, 4/4, 3/5 \rightarrow \frac{9!}{2! \cdot 4! \cdot 3!} = 1260$
 - $2/2, 3/4, 4/5 \rightarrow \frac{9!}{2! \cdot 3! \cdot 4!} = 1260$
 - $2/2, 2/4, 5/5 \rightarrow \frac{9!}{2! \cdot 2! \cdot 5!} = 756$
 - $1/2, 4/4, 4/5 \rightarrow \frac{9!}{1! \cdot 4! \cdot 4!} = 630$
 - $1/2, 3/4, 5/5 \rightarrow \frac{9!}{1! \cdot 3! \cdot 5!} = 504$
 - $0/2, 4/4, 5/5 \rightarrow \frac{9!}{0! \cdot 4! \cdot 5!} = 126$
- For a, we just add it all up.
- (a) $W = 4536$ ways
- For b, we exclude the one with 0 Passengers (0/2). We can just subtract 126 from (a)
- (b) $W = 4410$ ways

V) Common Models of Probability

Probability, p

- The ratio of successful outcomes (s) to all possible outcomes or events (n) which is represented as a fraction, decimal or percentage.

Simple Event

- Can be solved easily by listings
- Formulas
 - $p = \frac{s}{n}$
 - Successful outcomes
 - $q = \frac{f}{n} = 1 - p$
 - Failed outcomes
 - $p < 1$
 - $p + q = 1$
 - Odds in favor = $\frac{s}{f} = \frac{p}{q}$
 - Odds against = $\frac{f}{s} = \frac{q}{p}$

Example

A die is tossed once, what is the probability of getting 5?

- List of outcomes:
 - 1
 - 2
 - 3
 - 4
 - 5
 - Only this one is a successful outcome, $s = 1$
 - 6
- There are 6 outcomes. $n = 6$
- $P = \frac{1}{6}$

Example

A political analyst asked a group of people how they felt about two political policy statements. Each person was to respond A (agree), N (neutral), and D (disagree). Assuming each response combination is equally likely, what is the probability that a person interviewed agrees w/ exactly one of the two policy statements?

- By fundamental principle of counting
 - $p = \frac{1 \cdot 2 + 2 \cdot 1}{3 \cdot 3} = \frac{4}{9}$

- By listing:
 - List goes
 - AA
 - AN - 1
 - AD - 1
 - NA - 1
 - NN
 - ND
 - DA - 1
 - DN
 - DD
 - $s = 4$
 - $n = 9$
 - $p = \frac{s}{n}$
 - $p = \frac{4}{9}$

Example

A pair of dice is tossed together or in succession. Find the probability of making 8.

- By principle of counting, $n = 6 \cdot 6 = 36$
- Listing successes
 - $2 + 6$
 - $6 + 2$
 - $3 + 5$
 - $5 + 3$
 - $4 + 4$
- $s = 5$
- $p = \frac{s}{n} = \frac{5}{36}$
- Successes can also be counted by getting the total of the permutation of each combination adding up to 9
 - $2, 6 = 2! = 2$
 - $3, 5 = 2! = 2$
 - $4, 4 = \frac{2!}{2!} = 1$
 - Multiplicity of 2 for "4"
 - $s = 2 + 2 + 1 = 5$

Example

Three dice are tossed. Find the probability of making 9.

- By principle of counting, $n = 6 \cdot 6 \cdot 6 = 216$
- Getting the permutation of each combination adding up to 9
 - $1,2,6 = 3! = 6$
 - $1,3,5 = 3! = 6$
 - $1,4,4 = \frac{3!}{2!} = 3$
 - Multiplicity of 2 for “4”
 - $2,2,5 = \frac{3!}{2!} = 3$
 - Multiplicity of 2
 - $2,3,4 = 3! = 6$
 - $3,3,3 = \frac{3!}{3!} = 1$
- In making combination, list the elements in increasing order to make sure there will be no repetition of combination.
- $s = 6 + 6 + 3 + 3 + 6 + 1 = 25$
- $p = \frac{s}{n}$
- $p = \frac{25}{216}$

Example

A die is loaded in such a way that an even number is twice to occur as an odd number. Find the probability of getting less than 4.

- Listing
 - 1 - 1
 - 2 - 1
 - 2 - 1
 - 3 - 1
 - 4
 - 4
 - 5
 - 6
 - 6
- $s = 4$
- $n = 9$
- $p = \frac{s}{n}$
- $p = \frac{4}{9}$

- Another method:

- $x = \text{chance of odd no.}$
- $2x = \text{chance of even no.}$
- Listing:
 - $1 = x$ “success”
 - $2 = 2x$ “success”
 - $3 = x$ “success”
 - $4 = 2x$
 - $5 = x$
 - $6 = 2x$
- $p = \frac{s}{n} = \frac{x+2x+x}{x+2x+x+2x+x+2x} = \frac{4x}{9x}$
- $p = \frac{4}{9}$

Compound Events Probability

Let A and B be events.

A or B probability “Additive Rule”

$$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

(In set notations)

A and B probability “Multiplicative Rule”

“Events A and B happens together or in succession”

$$p(A \text{ and } B) = p(A) \times p(B) \quad \text{w/ order}$$

$$p(A \text{ and } B) = p(A) \times p(B) \times 2 \quad \text{w/o order}$$

Note: $p(A \text{ and } B) = p(B \text{ and } A)$

Not A “Complementary Rule”

$$p(\overline{A}) = 1 - p(A)$$

$$p(A) + p(\overline{A}) = 1$$

A given B “Conditional Probability”

$$p(A|B) = \frac{p(A \text{ and } B)}{p(B)}$$

Binomial Distribution Probability

“Answers how many successes in a certain number of trials.”

$$p(r) = nCr(p)^r(1 - p)^{n-r}$$

“n is fixed or constant”

- Where:
 - $p(r)$ = probability of r successes in n trials
 - p = probability of an event in one trial
 - For a coin to be heads or tails,
 $p = \frac{1}{2}, q = \frac{1}{2}$
 - For a die to be rolled to a certain number, $p = \frac{1}{6}, q = \frac{5}{6}$

$$q = 1 - p$$

Mean

$$\mu = np$$

Standard deviation

$$\sigma = \sqrt{npq}$$

Variance

$$\sigma^2 = npq$$

Negative Binomial Distribution Probability

“Answers how many trials to get a certain number of successes.”

$$p(n) = (n - 1)C(r - 1)(p)^r(1 - p)^{n-r}$$

“r is fixed or constant”

- Where:
 - $p(n)$ = probability that in n trials, there will be r successes
 - Or probability that n-th trial will be r-th success
 - p = probability of an event in one trial

Multinomial Distribution Probability

“Like partition formula where not each hole or spot has the same probability of being filled”

$$p = \frac{n!}{r_1! \cdot r_2! \cdot r_3! \cdot \dots} (p_1^{r_1} \cdot p_2^{r_2} \cdot p_3^{r_3} \cdot \dots)$$

- $n = r_1 + r_2 + r_3 + \dots$
- p_k = probability that event k will happen
- p = probability that event 1 occurs r_1 times AND probability that event 2 occurs r_2 times AND AND so on...

Hypergeometric Distribution Probability

“For probability of combinations or permutations of grouped elements”

“Formula similar to Partition Rule”

Without Order, No Replacement

$$p = \frac{\Pi nCr}{n_T Cr_T}$$
$$p = \frac{n_1 Cr_1 \cdot n_2 Cr_2 \cdot n_3 Cr_3 \cdot \dots}{n_T Cr_T}$$

With Order, No Replacement

$$p = \frac{\Pi nPr}{n_T Pr_T}$$
$$p = \frac{n_1 Pr_1 \cdot n_2 Pr_2 \cdot n_3 Pr_3 \cdot \dots}{n_T Pr_T}$$

- Where:
 - $n_T = n_1 + n_2 + n_3 + \dots$
 - $r_T = r_1 + r_2 + r_3 + \dots$

Poisson's Distribution Probability

“For probability of elements with known rate of occurrence”

“For probability that a number of discrete or individual events will happen in a given period of time”

Commonly applied to:

Communication and traffic
Printing errors
Waiting Lines

Can approximate:

People arriving in an hour
Phone calls in a day
Print jobs in a minute

$$p = \frac{e^{-\lambda} \lambda^x}{x!}$$

- Where:
 - $\lambda = rt$
 - λ = shape parameter which indicates the average number of events in the given time interval
 - r = rate which is the number of events per time period
 - t = time
 - x = no. of desired elements

Mean

$$\mu = \lambda$$

Standard deviation and Variance

$$\sigma = \sqrt{\lambda}, \quad \sigma^2 = \lambda$$

Negative Exponential Distribution Probability

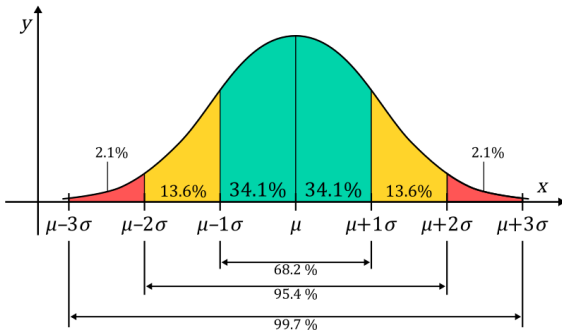
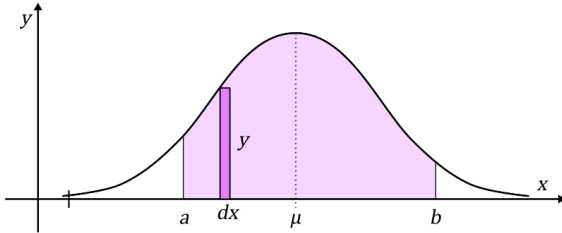
“Reliability”

- It is usually applied to reliability of an equipment between failures.

$$p = e^{-t/\mu}$$

- Where:
 - t = operating time
 - μ = mean time between failures

Normal Distribution Probability



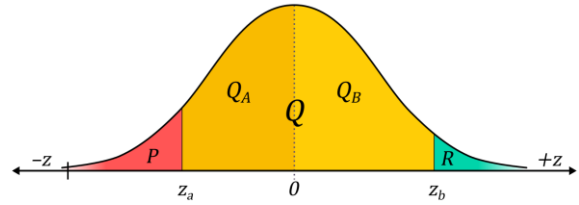
$$p(a < x < b) = \int_a^b y dx$$

- The probability is the area under normal curve:
 - $y = e^{-\frac{1}{2} \left(\frac{x-\mu}{\sigma} \right)^2} / \sigma \sqrt{2\pi}$
 - (Introduced by Gauss)

Standard Normal Distribution

“In terms of unit standard random variable or score z or t ”

$$z = \frac{x - \mu}{\sigma}$$



$$P(z_a) = p(z < z_a)$$

$$Q(z_a) = p(z_a < z < 0)$$

$$Q(z_b) = p(0 < z < z_b)$$

$$Z(z_b) = p(z > z_b)$$

$$Q = Q_A + Q_B$$

Probability with Uncountable Uniform Spaces

$$p = \frac{A_s}{A_n}$$

- Successful area over total area

Example

A card is drawn from a deck of playing cards. What is the probability that:

- It is a heart or a king?
- It is a king or an ace?

Recall:

- A deck of playing cards has 52 cards.
- 4 suits:
 - Diamond, D (red, R)
 - Heart, H (red, R)
 - Club, C (black, B)
 - Spade, S (black, B)
- A suit has 13 cards
- Face cards
 - Jack (J)
 - Queen (Q)
 - King (K)
- Numbered cards
 - 1 to 10
 - 1 is also Ace (A)

Solving...

- Use “Additive Rule”
- $p(H) = \frac{13}{52}$
- $p(K) = \frac{4}{52}$
- $p(H \text{ and } K) = \frac{1}{52}$
- $p(K \text{ or } K) = p(H) + p(K) - p(H \text{ and } K)$
- $p(H \text{ or } K) = \frac{13+4-1}{52}$
- **(a) $p(H \text{ or } K) = \frac{4}{13}$**
- $p(A) = \frac{4}{52}$
- $p(K \text{ and } A) = 0$
- $p(K \text{ or } A) = p(K) + p(A) - p(K \text{ and } A)$
- $p(K \text{ or } A) = \frac{4+4-0}{52}$
- **(b) $p(K \text{ or } A) = \frac{2}{13}$**

Example

The probability that a student passes Mathematics is $\frac{2}{3}$ and the probability that he passes English is $\frac{4}{9}$. If the probability of passing at least one subject is $\frac{4}{5}$, what is the probability that he will pass both subjects?

- Use “Additive Rule”
- “passing at least one subject” means either one is passed or both, $P(M \text{ or } E) = \frac{4}{5}$
- $P(E) = \frac{4}{9}$
- $P(M) = \frac{2}{3}$
- $P(M \text{ or } E) = p(M) + p(E) - p(M \text{ and } E)$
- $\frac{4}{5} = \frac{2}{3} + \frac{4}{9} - p(M \text{ and } E)$
- **$p(M \text{ and } E) = \frac{14}{45}$**

Example

Two cards are drawn in succession from a deck without replacement. What is the probability that both cards are greater than 3 but less than 8?

- Use “Multiplicative Rule”
- Cards with 4, 5, 6, 7 satisfy the condition. There are four suits of these four number cards. There are 16 cards in total satisfying the condition.
- Let:
 - C_1 : Event that drawn first card satisfies condition
 - C_2 : Event that drawn second card satisfies condition

- $P(C_1) = \frac{s}{n} = \frac{16}{52}$
- $P(C_2) = \frac{s}{n} = \frac{15}{51}$
 - If C_1 has a successful draw, and the cards drawn are not replaced, then there are only 15 successful outcomes and 51 total outcomes for the C_2 .
- $P(C_1 \text{ and } C_2) = P(C_1) \times P(C_2) = \frac{16}{52} \cdot \frac{15}{51}$
- $P(C_1 \text{ and } C_2) = \frac{20}{221}$

Example

Three cards are drawn without replacement from a deck of playing cards. Find the probability that the 1st card is a red ace, the 2nd card is 10 or Jack and the 3rd card is greater than 3 and less than 7.

- Use “Multiplicative Rule”
- $P(C_1) = \frac{2}{52}$
 - Two suits are red, so there are two red aces.
- $P(C_2) = \frac{4+4-0}{51} = \frac{8}{51}$
 - $P(C_2) = P(10) + P(J) - P(10 \text{ and } J)$
 - $n = 51$ since there is no replacement and one card is already drawn
- $P(C_3) = \frac{12}{50}$
 - 4, 5, 6 satisfies condition.
 - There are four suits of these three number cards. There are 12 successful outcomes in total.
 - $s = 12$
 - $n = 50$ since there is no replacement and two cards are already drawn
- $P(C_1 \text{ and } C_2 \text{ and } C_3) = P(C_1) \cdot P(C_2) \cdot P(C_3)$
- $P(C_1 \text{ and } C_2 \text{ and } C_3) = \frac{2}{52} \cdot \frac{8}{51} \cdot \frac{12}{50}$
- **$P(C_1 \text{ and } C_2 \text{ and } C_3) = \frac{8}{5525} = 0.00145$**

Example

A bag contains 4 red balls, 5 white balls, and 3 green balls. Find the probability of drawing a non-red ball.

- Use “Complementary Rule”
- $P(R) = \frac{s}{n} = \frac{4}{4+5+3} = \frac{4}{12}$
- $P(\overline{R}) = 1 - \frac{4}{12}$
- **$P(\overline{R}) = \frac{2}{3}$**

Example

A real estate agent has 8 master keys to open several new homes. Only one master key will open any given homes. If 40% of these homes are usually left unlocked, what is the probability that the real estate agent can get into a specific home if the agent selects 3 master keys at random before leaving the office?

- Use “Complementary Rule”
- $P(\text{can get to a home}) = 1 - P(\text{cannot get to home})$
- $P(\text{can get to a home}) = 1 - P(\text{took the wrong keys and door is locked})$
- $P(\text{can get to a home}) = 1 - P(\text{took the wrong keys}) \cdot P(\text{door is locked})$
- $P(\text{took the wrong keys}) = 1 - P(\text{took the right keys})$
- $P(\text{took the wrong keys}) = 1 - \frac{{}^1C_1 \cdot {}^7C_2}{{}^8C_3} = \frac{5}{8}$
 - 1 choose 1
 - 1 key is appropriate key and that 1 key is chosen
 - 7 choose 2
 - 7 key is remaining and 2 more keys is chosen so that 3 keys are taken in total
 - 8 choose 3
 - In 8 total keys a, a combination of 3 is chosen.
- $P(\text{door is locked}) = 1 - P(\text{door is unlocked})$
- $P(\text{door is locked}) = 1 - 0.4 = 0.6$
- $P(\text{can get to a home}) = 1 - \frac{5}{8}(0.6)$
- $P(\text{can get to a home}) = 1 - \frac{5}{8}(0.6)$
- **$P(\text{can get to a home}) = \frac{5}{8}$**

(Alternative solution)

- Use “Additive Rule”
- K : took right keys
- $\bar{K}$: took wrong keys
- D : door is open
- $\bar{D}$: door is locked
- Agent can enter the home if:
 - K and D OR...
 - K and $\bar{D}$ OR...
 - $\bar{K}$ and D

- $p = p(K \text{ and } D) + p(K \text{ and } \bar{D}) + p(\bar{K} \text{ and } D)$
- $p = p(K) + p(\bar{K}) \cdot p(D)$
- $p(K) = \frac{{}^1C_1 \cdot {}^7C_2}{{}^8C_3} = \frac{3}{8}$
- $p(\bar{K}) = 1 - \frac{3}{8} = \frac{5}{8}$
- $p(D) = 0.4$
- $p(\bar{D}) = 1 - 0.4 = 0.6$
- $p = \frac{3}{8} + \frac{5}{8}(0.4)$
- **$p = \frac{5}{8}$**

Example

The probability that a married man watches a certain TV show is 0.4 and the probability that a married woman watches the show is 0.5. If the probability that a man watches the show given that his wife does is 0.7, find the probability that:

- (a) A married couple watches the show?
- (b) The wife watches the show given that her husband does?

- Use “Conditional Probability”
- M : married man watches the show
- W : married woman watches the show
- $P(M|W) = 0.7$
- $P(M) = 0.4$
- $P(W) = 0.5$
- $P(M|W) = \frac{P(M \text{ and } W)}{P(W)}$
- $0.7 = \frac{P(M \text{ and } W)}{0.5}$
- **(a) $P(M \text{ and } W) = 0.35$**
- $P(W|M) = \frac{P(W \text{ and } M)}{P(M)} = \frac{0.35}{0.4}$
- **(b) $P(W|M) = 0.875$**

Example

A card is drawn from ordinary deck of cards and we are told that it is red. What is the probability that the card is greater than 2 but less than 9?

- Use “Conditional Probability”
 - Given that card is red
- $C: 2 < C < 9$
- R : card is red
- $2 < C < 9 = \{3, 4, 5, 6, 7, 8\}$

- 12 cards are red and are also greater than 2 but less than 9.
 - 2 suits are red.
 - $6 \times 2 = 12$
- 26 cards are red.
 - Half of a deck is red.
- “We are told that it is red” means event R is given
- $P(C|R) = \frac{P(C \text{ and } R)}{P(R)} = \frac{12/52}{26/52}$
- $P(C|R) = \frac{6}{13}$

Example

A coin is tossed 10 times. Find the probability of getting 3 heads.

- Use “Binomial Distribution”
- $n = 10$
- $r = 3$
- $p = \frac{1}{2}$
- $p(r) = nCr(p)^r(1-p)^{n-r}$
- $p(3) = 10C3\left(\frac{1}{2}\right)^3\left(1-\frac{1}{2}\right)^{10-3}$
- $p(3) = \frac{15}{128} = 0.1172$

Example

The probability that a baby hearing his mother’s voice on the 1st month is 90%. If 20 children are randomly picked:

- What is the probability that exactly 20 children hear their mother’s voice?
- What is the probability that at least 3 will not hear their mother’s voice?
- What is the mean and standard deviation?

- Use “Binomial Distribution”
- For a
 - $n = 20$
 - $r = 20$
 - $p = 90\%$
 - Probability that a baby will hear
 - $p(r) = nCr(p)^r(1-p)^{n-r}$
 - $p(20) = 20C20(0.9)^{20}(1-0.9)^{20-20}$
 - $p(20) = 0.1216$

- For b
 - $n = 20$
 - $r = 3 \text{ or } 4 \text{ or } 5 \text{ or } \dots 20$
 - “At least 3 will not hear”
 - 3 or more
 - $p = 10\% = 0.1$
 - Probability that will not hear
 - $p = \sum_{r=3}^{20} p(r)$
 - $p = \sum_{r=3}^{20} [20Cr(0.1)^r(1-0.1)^{20-r}]$
 - $p = 0.3231$
- For c
 - Use p of original problem or as given in the question.
 - $p = 0.9$
 - $q = 1 - p = 0.1$
 - $\mu = np = 20 \times 0.9 = 18$
 - $\sigma = \sqrt{npq} = \sqrt{20(0.9)(0.1)} = 1.34$
 - $\mu = 18 \text{ and } \sigma = 1.34$

Example

The probability that a person leaving in a city owns a dog is estimated to be 0.3. Find the probability that the 10th person randomly interviewed in the city is the 5th one to own a dog.

- Use “Negative Binomial Distribution”
- $p = 0.3$
- $n = 10$
- $r = 5$
- $p(n) = (n-1)C(r-1)(p)^r(1-p)^{n-r}$
- $p(10) = 9C4(0.3)^5(1-0.3)^{10-5}$
- $p(10) = 0.0515$

Example

According to the theory of genetics, a certain cross of guinea pig will result in red, black, and white offspring in the ratio 8:4:4. Find the probability that among 8 offspring, 5 will be red, 2 will be black, and 1 will be white.

- Use “Multinomial Distribution”
- $r_1 = 5$
- $r_2 = 2$
- $r_3 = 1$
- $n = r_1 + r_2 + r_3 = 8$

- $p_1 = \frac{8}{8+4+4} = \frac{1}{2}$
- $p_2 = \frac{4}{8+4+4} = \frac{1}{4}$
- $p_3 = \frac{4}{8+4+4} = \frac{1}{4}$
- $p = \frac{n!}{r_1! \cdot r_2! \cdot r_3!} (p_1^{r_1} \cdot p_2^{r_2} \cdot p_3^{r_3})$
- $p = \frac{8!}{5! \cdot 2! \cdot 1!} \left(\frac{1}{2}\right)^5 \left(\frac{1}{4}\right)^2 \left(\frac{1}{4}\right)$
- $p = \frac{21}{256} = 0.0820$

Example

A homeowner plants 6 bulbs selected at random from a box containing 5 tulip and 4 daffodil bulbs. What is the probability that he planted 2 daffodil bulbs and 4 tulip bulbs?

- Use “Hypergeometric Distribution”
- 1: tulip bulbs
- 2: daffodil bulbs
- $n_1 = 5$
- $n_2 = 4$
- $n_T = n_1 + n_2 = 9$
- $r_1 = 4$
- $r_2 = 2$
- $r_T = r_1 + r_2 = 6$
- $p = \frac{\Pi nCr}{n_T Cr_T} = \frac{5C4 \cdot 4C2}{9C6}$
- $p = \frac{5}{14}$

Example

In a poker hand consisting of 5 cards, find the probability of holding:

- (a) 3 aces
- (b) 4 hearts and 1 club
- Use “Hypergeometric Distribution”
- For a
 - Groups are ace cards (1) and non-ace cards (2)
 - $n_T = 52, r_T = 5$
 - $n_1 = 4, r_1 = 3$
 - $n_2 = 48, r_2 = 2$
 - $p = \frac{\Pi nCr}{nCr} = \frac{4C3 \cdot 48C2}{52C5}$

$$\circ p = \frac{94}{54145} = 0.0017$$

- For b
 - Groups are heart cards (1) and club cards (2) and non-heart and non-club cards (3)
 - $n_T = 52, r_T = 5$
 - $n_1 = 13, r_1 = 4$
 - $n_2 = 13, r_2 = 1$
 - $n_3 = 26, r_3 = 0$
 - $p = \frac{\Pi nCr}{nCr} = \frac{13C4 \cdot 13C1 \cdot 26C0}{52C5}$
 - $p = \frac{143}{39984} = 0.0036$

Example

On the average, a certain intersection results 3 traffic accidents per month. What is the probability that in any given month at this intersection, exactly 5 accidents will occur?

- Use Poisson’s distribution probability since there is a known rate $r = 3$
- $t = 1$
- $\lambda = rt = 3$
- $x = 5$
- $p = \frac{e^{-\lambda} \lambda^x}{x!}$
- $p = \frac{e^{-rt} (rt)^x}{x!} = \frac{e^{-3} (3)^5}{5!}$
- $p = 0.1008$

Example

Suppose that you are watching people arriving at a doctor’s office and they are arriving at an average of 3 per hr. What is the probability that in a certain hour no one will arrive?

- Use Poisson’ distribution probability since there is a known rate $r = 3$
- $t = 1$
- $\lambda = rt = 3$
- $x = 0$
- $p = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-3} (3)^0}{0!}$
- $p = 0.0498$

Example

Suppose 300 misprints are distributed randomly throughout a book of 500 pages. Find the probability that a given page contains exactly 2 misprints.

- Use Poisson's distribution probability since a rate r misprints per page can be computed.
- $r = \frac{300}{500} = \frac{3}{5}$ misprint per page
- $t = 1$ page
- $\lambda = \frac{3}{5}$
- $x = 2$
- $p = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-3/5} (3/5)^2}{2!}$
- **$p = 0.0988$**

Example

Heavy equipment exhibits a negative exponential failure distribution w/ a mean time to failure of 1000 hrs. Find the maximum operating time such that the reliability remains 99%.

- Use Negative Exponential Distribution Probability
- $\mu = 1000$ hrs
- $t = ?$
- $p = 99\% = 0.99$
- $p = e^{-t/\mu} =$
- $0.99 = e^{-t/1000}$
- **$t = 10.05$ hrs**

Example

A certain type of storage battery last on the average 3.0 years with standard deviation of 0.5 years. Assuming that the battery lives are normally distributed, find the probability that a given battery will last less than 2.3 years.

- Use normal distribution as given in problem.
- $\mu = 3$
- $\sigma = 0.5$
- $p(x < 2.3) = ?$
- $z = \frac{x-\mu}{\sigma} = \frac{2.3-3}{0.5} = -1.4$
- $p(x < 2.3) = p(z < -1.4) = P(-1.4)$
- **$p(x < 2.3) = 0.80757$**

Example

An electrical firm manufactures light bulbs that have a length of life that is normally distributed with mean equal to 800 hrs. and a standard deviation of 40 hrs. Find the probability that the bulb burns between 778 and 834 hours.

- Use normal distribution as given in problem.
- $\mu = 800$
- $\sigma = 40$
- $p(778 < x < 834) = ?$
- $z = \frac{x-\mu}{\sigma}$
- $z_1 = \frac{778-800}{40} = -0.55$
- $z_2 = \frac{834-800}{40} = 0.85$
- $p(778 < x < 834) = p(-0.55 < z < 0.85)$
- $p(778 < x < 834) = Q(-0.55) + Q(0.85)$
- $p(778 < x < 834)$
 - $= Q\left(\frac{778-800}{40}\right) + Q\left(\frac{834-800}{40}\right)$
- **$p(778 < x < 834) = 0.51118$**

Example

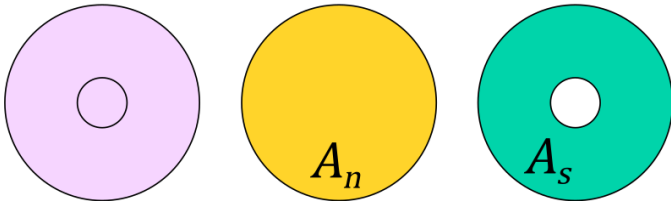
The mean duration of television commercials on a given network is 75 seconds, with standard deviation of 20 seconds. Assume that duration times are normally distributed, what is the probability that a commercial will last less than 35 seconds?

- Use normal distribution as given in problem.
- $\mu = 75$
- $\sigma = 20$
- $p(x < 35) = ?$
- $z = \frac{x-\mu}{\sigma} = \frac{35-75}{20} = -2$
- $p(x < 35) = p(z < -2) = P(-2)$
- **$p(x < 35) = 0.02275$**

Example

A point is selected at random inside a circle with diameter 8 cm. Find the probability that the point is at least 1.5 cm away from the center.

- Use “Probability with Uncountable Uniform Spaces”
- “At least 1.5 cm away from center” means it can 1.5 cm away or more, or the point is at or farther than the 1.5 cm radius circle.
- Sketching...



- $p = \frac{A_s}{A_n} = \frac{\pi(4^2 - 1.5^2)}{\pi(4^2)}$
 - Diameter was divided by 2 to get radius of 4 cm.

$$p = \frac{55}{64} = 0.8594$$

- When coin is pushed to the edge of the square, the center of the coin is on the boundary of the winning area.
- Closer to the center of the square, it is a win.
- Farther from the center of the square, it is a lose because the coin will cross the side of the square already.
- Probability of losing is the required.
- S : side of big square
- s : side of small square
- $s = 2(0.5 - r) = 2\left(0.5 - \frac{0.75}{2}\right) = 0.25$
- $S = 1$
- $p = \frac{A_s}{A_n} = \frac{S^2 - s^2}{S^2} = \frac{1^2 - 0.25^2}{1^2}$
- $p = \frac{15}{16} = 0.9375$

Example

A coin is thrown inside a 1 inch by 1 inch square and wins if exactly inside the square and lose if the coin crosses the side of the square. What is the probability of losing if the diameter of coin is $\frac{3}{4}$ inch?

- Use “Probability with Uncountable Uniform Spaces”
- Sketching...

